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Research Paper

A model for the effect of toxicant on a three species food-chain system with "food-limited" growth of prey population

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Abstract

In this paper, a mathematical model is proposed and analyzed to study the effect of toxicant in a three species food chain system with "food-limited" growth of prey population. The mathematical model is formulated using the system of non-linear ordinary differential equations. In the model, there are seven state variables, viz, prey density, intermediate predator density, density of top predator, concentration of toxicant in the environment, concentration of toxicant in the prey, concentration of toxicant in the intermediate predator and concentration of toxicant in the top predator. In the model, it is assumed that the carrying capacity and growth rate of prey is affected by environmental toxicant. Toxicant is transferred to intermediate predator and top predator populations through food chain pathways. All the feasible equilibria of the system are obtained and the conditions are determined for the survival or extinction of species under the effect of toxicant. The local and global stability analysis of all the feasible equilibria are carried out. Further, the results are compared with the case when toxicant is absent in the system. Finally, we support our analytical findings with numerical simulations.

Keywords: *Food chain; Toxicant; Food-Limited Growth; Stability; Lyapunov function.*

1. Introduction

The effects of toxic substances on ecological communities is an important challenging problem from an environmental point of view. Many species are exposed to various kinds of stresses including toxicants which are affecting their growth rate, carrying capacity and their resources. The effects of toxicants on ecological communities including three-species food chain systems are very complex dynamical systems to be undertaken for mathematical study.

Generally a single population grows logistically and implicit assumption contained in the logistic growth equation is that the average growth rate is a linear function of the population density. It has been shown that this assumption is not realistic for a population with food-limited growth. Filter feeders strain the water column indiscriminately for small prey, typically phyto- and zooplankton. This category of fishes includes threadfin shad, American shad, inland silversides and anchovies. Some evidence suggests that some of these species are "food-limited" due to the depressed levels of plankton after the introduction of the Amur River clam [1]. David [2] in his experimental study, says that the labyrinth spider appears to be a "food-limited" species in which exploitative competition for food is weak or absent. It has been found in the nature that the predators for spider are mouse and lizard and mouse is predated by snake whereas the lizards are killed and eaten by hawk.

The food-limited population model incorporated the concept of limited food and space and it was formulated by modifying the logistic growth equation considering the average growth rate to be a non-linear function of population density. The food-limited population models have been proposed by several researchers [3, 4] for the dynamics of a population where growth limitations were based on the proportion of available resources not utilized. Some studies about food-limited population models have been carried out by few authors [5, 6], who have obtained an interesting results about the stability and Hopf bifurcation of positive solutions.

Three species food chain systems have received much attention from many applied mathematicians and ecologists in recent years [7, 8]. In [9], Zhang et.al, studied and established an experimental marine food chain of three levels ($microalgae \rightarrow zooplankton \rightarrow fish$) to investigate the effect of feeding selectivity on the transfer of methyl mercury (MeHg) through the food chain system.

Fish-eating birds in certain parts of the United States may ingest large amounts of methylmercury in their diet [10]. The methylmercury-containing bacteria may be consumed by the next higher level in the food chain, or the bacteria may excrete the methylmercury to the water where it can quickly adsorb to plankton, which are also consumed by the next level in the food chain [11, 12]. Because animals accumulate methylmercury faster than they eliminate it, animals consume higher concentrations of mercury at each successive level of the food chain. Small environmental concentrations of methylmercury can thus readily accumulate to potentially harmful concentrations in fish, fish-eating wildlife and people. Even at very low atmospheric deposition rates in locations remote from point sources, mercury biomagnification can result in toxic effects in consumers at the top of these aquatic food chains. Poisoning from pesticides can travel up the food chain [13, 14]; for example, birds can be harmed when they eat insects and worms that have consumed pesticides. A number of studies [15, 16] have shown that pesticides have had harmful effects on growth and reproduction of earthworms, which are in turn consumed by terrestrial vertebrates such as birds and small mammals.

Previously, some research have been done on tri-trophic food-chain systems including toxicant effects on the survival or extinction of species in the system [17, 18]. It has been observed that toxicants have very pronounced effects on the species if the availability of the resources is limited. To our knowledge, almost no studies have been conducted to investigate the effect of toxicant on a three-species food chain systems with "food-limited" growth of prey population and therefore, in this paper, a mathematical model is proposed to study the effects of toxicants on a three species food-chain system with "food-limited" growth of prey population. The present model may be suited for the food chain system comprising of $Spider \rightarrow Mouse \rightarrow Snake$ and also for the food chain system consisting of $Spider \rightarrow Lizard \rightarrow Hawk$.

2. Mathematical model

We have considered a three species food chain system under the stress of a toxicant considering "food-limited" growth of prey population. In the model, it is assumed that the growth rate and the carrying capacity of prey is negatively affected by environmental toxicant [19]. Toxicant is transferred to intermediate predator and top predator populations through food chain pathways. For the prey population, a simple "food-limited" growth equation [3, 4], is considered. Lotka-Volterra type of prey-predator interaction is considered in the model. The model is formulated with the help of following system of ordinary differential equations

Main Model: (With toxic effect)

$$\frac{dx}{dt} = xr(U)\left(\frac{K(T)-x}{K(T)+r_0cx}\right) - a_1xy \quad (1)$$

$$\frac{dy}{dt} = \beta_1a_1xy - a_2yz - \beta_{11}Vy - d_1y - b_1y^2 \quad (2)$$

$$\frac{dz}{dt} = \beta_2a_2yz - \beta_{22}Wz - d_2z - c_3z^2 \quad (3)$$

$$\frac{dT}{dt} = Q_0 - \delta_0T - \alpha_1xT \quad (4)$$

$$\frac{dU}{dt} = \alpha_1xT - \delta_1U - \beta_3(U)a_1xy \quad (5)$$

$$\frac{dV}{dt} = \beta_3(U)a_1xy - \delta_2V - \beta_4(V)a_2yz \quad (6)$$

$$\frac{dW}{dt} = \beta_4(V)a_2yz - \delta_3W \quad (7)$$

The above system of ordinary differential equations are associated with the following initial conditions:

$$x(0) > 0, y(0) > 0, z(0) > 0, T(0) > 0, U(0) \geq 0, V(0) \geq 0, W(0) \geq 0.$$

In the above model, x is the density of prey population, y is the density of intermediate predator population, z is the density of top predator population, T is the concentration of toxicant in the environment, U is the concentration of toxicant in the prey population, V is the concentration of toxicant in the intermediate predator population and W is the concentration of toxicant in the top predator population. In the model, first term in the prey equation describes "food-limited" growth rate function under the effect of toxicant, d_1 and d_2 are the death rates of intermediate predator and top predator populations respectively, b_1 and c_3 are the intraspecific competition rates due to crowding of intermediate and top predator populations respectively. $K(T)$ represents the carrying capacity of prey which is negatively affected by T, the function $r(U)$ denotes the specific growth rate of prey population which is negatively affected by U, β_1 and β_2 are conversion coefficients, $\beta_3(U)$ and $\beta_4(V)$ are toxicant transfer functions. Q_0 is the rate of introduction of toxicant into the environment. δ_0, δ_1, δ_2 and δ_3 are first order decay rates of toxicants in the environment as well as in the populations. β_{11} and β_{22} are the death rates of predators due to organismal toxicant concentration. k_0 is the natural carrying capacity. Food limited parameter is denoted by c. r_0 is the intrinsic growth rate of prey. k_1 and r_1 are the constants which determine the rate of decrease of carrying capacity and growth rate of prey population respectively due to the presence of toxicant. α_1 is the depletion rate of toxicant in the environment due to its intake by the population. a_1 and a_2 are the loss of prey and intermediate predator due to intermediate and top predators. c, a_1, a_2, a_3, a_4 and α_1 are the positive constants.

For our analysis, in the model we consider, $K(T) = k_0 - k_1 T$, $r(U) = r_0 - r_1 U$, $\beta_3(U) = a_3 U$ and $\beta_4(V) = a_4 V$. Now in order to compare the results of main model with the system which is free from toxicant, we analyze the following sub-system:

Sub Model: (Without toxic effect)

$$\frac{dx}{dt} = r_0 x\left(\frac{k_0 - x}{k_0 + r_0 c x}\right) - a_1 x y \tag{8}$$

$$\frac{dy}{dt} = a_1\beta_1 x y - a_2 y z - d_1 y - b_1 y^2 \tag{9}$$

$$\frac{dz}{dt} = a_2\beta_2 y z - d_2 z - c_3 z^2 \tag{10}$$

The above system of ordinary differential equations are associated with the following initial conditions: $x(0) > 0, y(0) > 0, z(0) > 0$. Where, the state variables and parameters are the same as defined for the main model.

3. Analysis of Sub-Model

The sub model has following four non-negative equilibria in x, y, z space namely, $\hat{E_{20}} = (0,0,0)$, $\check{E_{21}} = (k_0,0,0)$, $\bar{E_{22}} = (\bar{x}, \frac{a_1\beta_1\bar{x} - d_1}{b_1}, 0)$ is positive under conditions: $\bar{x} = \frac{-S_2 \pm \sqrt{S_2^2 + 4S_1S_3}}{2S_1} > 0$ and $a_1\beta_1\bar{x} > d_1$,
where, $S_1 = r_0 c a_1^2 \beta_1$, $S_2 = r_0 b_1 + a_1^2 k_0 \beta_1 - r_0 c a_1 d_1$, $S_3 = k_0(r_0 b_1 + a_1 d_1)$
and $E_{23}^* = (x^*, \frac{a_1\beta_1 c_3 x^* + (a_2 d_2 - d_1 c_3)}{b_1 c_3 + a_2^2\beta_2}, \frac{1}{c_3}(a_2\beta_2 y^* - d_2))$ is positive under the following conditions:
$x^* = \frac{-H_2 \pm \sqrt{H_2^2 + 4H_1H_3}}{2H_1} > 0$, clearly, $H_3 > 0$, $a_2 d_2 > d_1 c_3$ and $a_2\beta_2 y^* > d_2$,
where, $H_1 = r_0 c a_1^2 \beta_1 c_3$, $H_2 = a_1^2\beta_1 k_0 c_3 + r_0(b_1 c_3 + a_2^2\beta_2) + r_0 c a_1(a_2 d_2 - d_1 c_3)$, $H_3 = r_0 k_0(b_1 c_3 + a_2^2\beta_2) - a_1 k_0(a_2 d_2 - d_1 c_3)$. Now, we will discuss the dynamical behavior of the sub-model.

- The equilibrium point $\hat{E_{20}}$ is always unstable.
- The equilibrium point $\check{E_{21}}$ is locally asymptotically stable under condition: $a_1\beta_1 k_0 < d_1$.
  Remark 1: Here, it may be noted that if the product of the predation rate of intermediate predator, its conversion efficiency and carrying capacity of prey is less than the death rate of intermediate predator then prey population will survive and both the predator populations will go to extinction.
- The equilibrium point $\bar{E_{22}}$ is locally asymptotically stable under the following conditions:
  $a_2\beta_2\bar{y} < d_2$ and $a_1\beta_1\bar{x} > d_1$.
  Remark 2: Here, it may be noted that if the product of the predation rate of top predator, its conversion efficiency and equilibrium of intermediate predator is less than the death rate of top predator, and also

if the product of the predation rate of intermediate predator, its conversion efficiency and equilibrium of prey population is greater than the death rate of intermediate predator then prey and intermediate predator populations will survive and the top predator may tend to extinction.

- The equilibrium point $E^*_{23}$is locally asymptotically stable under the following conditions: $a_2\beta_2 y^* > d_2$ and $a_2 d_2 > d_1 c_3$.
  Remark 3: From the above conditions, it may be noted that if the product of the predation rate of top predator, its conversion efficiency and equilibrium of intermediate predator is greater than the death rate of top predator, and also if the product of the predation rate of top predator and its death rate is greater than the death rate of intermediate predator is multiplied by intraspecific competition rate due to crowding of top predator then all the population will survive.

Now, we will establish that the system described by Sub-Model is bounded. We begin with the following Lemma.
Lemma 3.2: The set $\Omega_2 = \{(x, y, z) : 0 \le x(t) \le k_0, 0 \le \beta_1 x(t) + y(t) + \frac{1}{\beta_2} z(t) \le w_1\}$ is a region of attraction for all solutions initiating in the interior of the positive region, where $w_1 = \frac{x_u \beta_1 (r_0 k_0 + 1)}{\Phi_1}$, $\Phi_1 = min\{1, d_1, d_2\}$.
Proof: From (8) we get,

$$\frac{dx}{dt} \le r_0 x(\frac{k_0 - x}{k_0 + r_0 c x})$$

then by the usual comparison theorem, we get as $t \to \infty$, $x \le k_0$.
Now, let us consider the following function: $w_1(t) = \beta_1 x(t) + y(t) + \frac{1}{\beta_2} z(t)$
by using (8) to (10), we get

$$\frac{dw_1}{dt} + \Phi_1 w_1 \le x_u \beta_1 (r_0 k_0 + 1)$$

where $\Phi_1 = min\{1, d_1, d_2\}$ then by the usual comparison theorem, we get as $t \to \infty$,

$$w_1 = \frac{x_u \beta_1 (r_0 k_0 + 1)}{\Phi_1}$$

This proves the lemma.
Theorem 3.2: If the following inequalities hold in the region Ω_2,

$$2a_1^2 D_1 (1 - \beta_1 y_u)^2 < r_0 k_0 (1 + r_0 c)(a_2(z_u - z^*) + b_1 y_u) \tag{11}$$

$$2a_2^2 (y^* - \beta_2 z_u)^2 < c_3 z_u (a_2(z_u - z^*) + b_1 y_u) \tag{12}$$

then the positive equilibrium E^*_{23} is globally asymptotically stable with respect to all solutions initiating in the interior of positive region Ω_2. Where $D_1 = (k_0 + r_0 c x_u)(k_0 + r_0 c x^*)$.

Proof: We consider the following positive definite function about E^*_{23}:

$$V_2 = (x - x^* - x^* ln(\tfrac{x}{x^*})) + \tfrac{I_1}{2}(y - y^*)^2 + \tfrac{I_2}{2}(z - z^*)^2$$

Differentiating V with respect to time t, we get

$$\frac{dV_2}{dt} = (\frac{x - x^*}{x})\frac{dx}{dt} + I_1 (y - y^*)\frac{dy}{dt} + I_2 (z - z^*)\frac{dz}{dt}$$

Using system of equations (8)-(10), we get after some algebraic manipulations

$$\begin{aligned} \frac{dV_2}{dt} &= -(x - x^*)^2 (1 + r_0 c)(r_0 k_0 / D_1) - (y - y^*)^2 (d_1 + a_2 z - a_1 \beta_1 x^* + b_1 (y + y^*))(I_2/2) \\ &\quad -(z - z^*)^2 (d_2 + c_3 (z + z^*) - a_2 \beta_2 y^*) I_2 - (x - x^*)(y - y^*) a_1 (1 - \beta_1 I_1 y) \\ &\quad -(y - y^*)(z - z^*) a_2 (I_1 y^* - \beta_2 I_2 z) \end{aligned}$$

Now, dV_2/dt can further be written as sum of the quadratic forms as

$$\begin{aligned} \frac{dV_2}{dt} &\le -[(\frac{1}{2} a_{11} (x - x^*)^2 + a_{12} (x - x^*)(y - y^*) + \frac{1}{2} a_{22} (y - y^*)^2) \\ &\quad +(\frac{1}{2} a_{22} (y - y^*)^2 + a_{23} (y - y^*)(z - z^*) + \frac{1}{2} a_{33} (z - z^*)^2)] \end{aligned}$$

where, $a_{11} = (1 + r_0c)(r_0k_0/D_1)$, $a_{12} = a_1(1 - \beta_1 I_1 y)$, $a_{22} = (I_1/2)(a_2(z - z^*) + b_1y^*)$, $a_{23} = a_2(I_1y^* - \beta_2 I_2 z)$, $a_{33} = I_2c_3z$, $D_1 = (k_0 + r_0cx)(k_0 + r_0cx^*)$. Now, by using Sylvester's criteria and by choosing $I_1 = I_2 = 1$, we get that dV_2/dt is negative definite under the following conditions:

$$a_{11}a_{22} > a_{12}^2 \tag{13}$$
$$a_{22}a_{33} > a_{23}^2 \tag{14}$$

We note that, (11) ⇒ (13) and (12) ⇒ (14). Hence V_2 is a Lyapunov function with respect to E_{23}^*, whose domain contains the region of attraction Ω_2, proving the theorem.

4. Analysis of Main Model

4.1. Equilibria of Main Model

The Main Model has four non negative equilibria in x, y, z, T, U, V, W space namely, $\hat{E_{10}} = (0, 0, 0, \hat{T}, 0, 0, 0)$, $\check{E_{11}} = (\check{x}, 0, 0, \check{T}, \check{U}, 0, 0)$, $\bar{E_{12}} = (\bar{x}, \bar{y}, 0, \bar{T}, \bar{U}, \bar{V}, 0)$ and $E_{13}^* = (x^*, y^*, z^*, T^*, U^*, V^*, W^*)$. The existence of $\hat{E_{10}}$ is obvious. We prove the existence of $\check{E_{11}}$, $\bar{E_{12}}$ and E_{13}^* as follows:

- $\hat{E_{10}}(0, 0, 0, \frac{Q_0}{\delta_0}, 0, 0, 0)$
- $\check{E_{11}}(\check{x}, 0, 0, \check{T}, \check{U}, 0, 0)$

$$\check{T} = \frac{Q_0}{\delta_0 + \alpha_1\check{x}} = f_1(x)$$
$$\check{U} = \frac{\alpha_1\check{x}\check{T}}{\delta_1} = \frac{\alpha_1\check{x}f_1(x)}{\delta_1} = f_2(x)$$

  and $\check{x}$ is given by the following quadratic equation: $A_1\check{x}^2 + A_2\check{x} - A_3 = 0$, where, $A_1 = \alpha_1$, $A_2 = \delta_0 - k_0\alpha_1$, $A_3 = k_0\delta_0 - k_1Q_0$. The equation will have a positive root provided $k_0\delta_0 > k_1Q_0$ holds good.

- $\bar{E_{12}}(\bar{x}, \bar{y}, 0, \bar{T}, \bar{U}, \bar{V}, 0)$

$$(r_0 - r_1U)\left(\frac{K(T)-x}{K(T)+r_0cx}\right) - a_1y = 0 \tag{15}$$
$$\beta_1a_1x - \beta_{11}V - d_1 - b_1y = 0 \tag{16}$$
$$Q_0 - \delta_0T - \alpha_1xT = 0 \tag{17}$$
$$\alpha_1xT - \delta_1U - a_1a_3Uxy = 0 \tag{18}$$
$$a_1a_3Uxy - \delta_2V = 0 \tag{19}$$

In this case, $\bar{x}$, $\bar{y}$, $\bar{T}$, $\bar{U}$ and $\bar{V}$ are the positive solutions of the system of equations from (17),

$$T = \frac{Q_0}{\delta_0 + \alpha_1x} = h_1(x) \tag{20}$$

by doing, $[\delta_2(16) - \beta_{11}(18 + 19)]$, we get,

$$U = \frac{d_1\delta_2 + \alpha_1\beta_{11}xh_1(x) - a_1\beta_1\delta_2 + \frac{r_0b_1\delta_2}{a_1}\left(\frac{K(h_1(x))-x}{K(h_1(x))+r_0cx}\right)}{\delta_1\beta_{11} + \frac{r_0b_1\delta_2}{a_1}\left(\frac{K(h_1(x))-x}{K(h_1(x))+r_0cx}\right)} = h_2(x) \tag{21}$$

from (15) and (16),

$$V = \frac{1}{a_1\beta_{11}}\left(a_1^2\beta_1x - a_1d_1 - b_1r(h_2(x))\left(\frac{K(h_1(x)) - x}{K(h_1(x)) + r_0cx}\right)\right) = h_3(x) \tag{22}$$

from (16) to (19),

$$y = \frac{1}{b_1\delta_1}(\delta_0\beta_{11}h_1(x) + \delta_1\beta_{11}h_2(x) + a_1\beta_1\delta_2x - d_1\delta_2 - Q_0\beta_{11}) = h_4(x) \tag{23}$$

Let

$$P(x) = a_1a_3xh_2(x)h_4(x) - \delta_2h_3(x) \tag{24}$$

Then we note that

$$P(0) = \frac{\delta_2}{a_1\beta_{11}}(a_1d_1 + b_1r(U)) > 0$$

and

$$P(k_0) = a_1a_3k_0h_2(k_0)h_4(k_0) - \delta_2h_3(k_0) < 0$$

This guarantees the existence of a root of $P(x) = 0$ for $0 < x < k_0$, say $\bar{x}$.
Further, this root will be unique provided

$$P'(x) = a_1a_3[h_2(x)h_4(x) + xh_2'(x)h_4(x) + xh_2(x)h_4'(x)] - \delta_2h_3'(x) < 0 \tag{25}$$

Knowing the value of $\bar{x}$, the values of $\bar{T}$, $\bar{U}$, $\bar{V}$ and $\bar{y}$ can be computed from equations (20) to (23) respectively.

- $E^*_{13}(x^*, y^*, z^*, T^*, U^*, V^*, W^*)$
  Here x^*, y^*, z^*, T^*, U^*, V^* and W^* are the positive solutions of the system of algebraic equations from (2),

$$T = \frac{Q_0}{\delta_0 + \alpha_1x} = g_1(x) \tag{26}$$

from (5),

$$U = \frac{\alpha_1xg_1(x)}{\delta_1 + a_1a_3xy} = g_2(x, y) \tag{27}$$

from (1) to (4), we get

$$z = \frac{\beta_1\beta_{22}a_1\delta_2x + yi_{11} - i_{22} - \beta_{11}\beta_{22}(Q_0 - \delta_0g_1(x) - \delta_1g_2(x, y))}{a_2\delta_2\beta_{22} + c_3\beta_{11}\delta_3} = g_3(x, y) \tag{28}$$

where, $i_{11} = \beta_{11}\beta_2a_2\delta_3 - b_1\delta_2\beta_{22}$, $i_{22} = d_1\delta_2\beta_{22} + d_2\delta_3\beta_{11}$.
from (6),

$$V = \frac{a_1a_3xyg_2(x, y)}{\delta_2 + a_2a_4yg_3(x, y)} = g_4(x, y) \tag{29}$$

from (3),

$$W = \frac{1}{\beta_{22}}(a_2\beta_2y - d_2 - c_3g_3(x, y)) = g_5(x, y) \tag{30}$$

Now, considering two functions,

$$G_{11}(x, y) = [r_0 - r_1g_2(x, y)](K(g_1(x)) - x) - a_1y[K(g_1(x)) + r_0cx] = 0 \tag{31}$$

$$G_{12}(x, y) = Q_0 - \delta_0g_1(x) - \delta_2g_4(x, y) - \delta_3g_5(x, y) = 0 \tag{32}$$

For existence of x^* and y^*, the two isoclines,

$$G_{11}(x, y) = 0 \tag{33}$$

$$G_{12}(x, y) = 0 \tag{34}$$

must intersect.
We note that

$$G_{11}(0, 0) = \frac{r_0}{\delta_0}(k_0\delta_0 - k_1Q_0) > 0$$

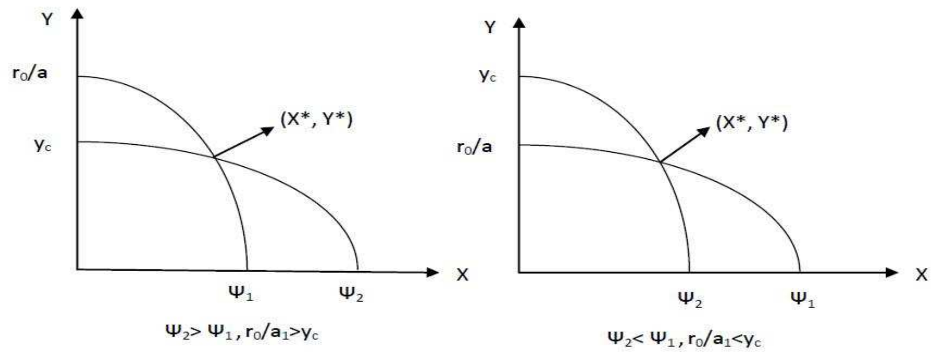

Fig.4.1

$G_{11}(0,0) > 0$ if $k_0\delta_0 > k_1Q_0$.

$$G_{12}(0,0) = \frac{\delta_2\delta_3(a_2d_2 - d_1c_3)}{a_2\delta_2\beta_{22} + \beta_{11}c_3\delta_3} > 0$$

$G_{12}(0,0) > 0$ if $a_2d_2 > d_1c_3$.
Also,
$G_{11}(0,y) = 0$ then $y = \frac{r_0}{a_1}$.
$G_{11}(x,0) = 0$ then x will have one positive root (ψ_1 say), from the following cubic equation of x,
$E_{11}x^3 + E_{12}x^2 + E_{13}x - E_4 = 0$, where, $E_{11} = \alpha_1^2(r_0\delta_1 - r_1Q_0)$, $E_{12} = \alpha_1[(r_0\delta_1 - r_1Q_0)(\delta_0 - \alpha_1k_0) + r_0\delta_0\delta_1]$, $E_{13} = r_0\delta_0\delta_1(\delta_0 - \alpha_1k_0) - \alpha_1(r_0\delta_1 - r_1Q_0)(k_0\delta_0 - k_1Q_0)$, $E_4 = r_0\delta_0\delta_1(k_0\delta_0 - k_1Q_0)$. Here, $E_{11} > 0$, $E_{12} > 0$, $E_{13} > 0$ and $E_4 > 0$. If

$$\delta_0 > \alpha_1k_0, r_0\delta_1 > r_1Q_0, k_0\delta_0 > k_1Q_0,$$

$$r_0\delta_0\delta_1(\delta_0 - \alpha_1k_0) > \alpha_1(r_0\delta_1 - r_1Q_0)(k_0\delta_0 - k_1Q_0).$$

$G_{12}(0,y) = 0$ then

$$y = \frac{\delta_2\beta_{22}(a_2d_2 - d_1c_3)(a_2\delta_2\beta_{22} + \beta_{11}c_3\delta_3)}{c_3(b_1\delta_2\beta_{22} - \beta_{11}\beta_2a_2\delta_3)} = y_c(say) > 0,$$

if $a_2d_2 > d_1c_3$, $b_1\delta_2\beta_{22} > \beta_{11}\beta_2a_2\delta_3$.
$G_{12}(x,0) = 0$ then x will have one positive root (ψ_2 say), from the following quadratic equation of x,
$J_1x^2 - J_2x - J_3 = 0$, where, $J_1 = a_1\alpha_1\delta_2c_3\delta_3\beta_1\beta_{22}$, $J_2 = \alpha_1\delta_2\delta_3\beta_{22}(a_2d_2 - d_1c_3) + \alpha_1\beta_{22}Q_0(a_2\delta_2\beta_{22} + \beta_{11}c_3\delta_3) - a_1\delta_2\delta_3c_3^2\beta_1\beta_{22}$, $J_3 = \delta_2\beta_{22}(a_2d_2 - d_1c_3)$. For x,

$$x = \frac{J_2 \pm \sqrt{J_2^2 + 4J_1J_3}}{2J_1} > 0$$

clearly, $J_3 > 0$ *i.e.*, $a_2d_2 > d_1c_3$.
Thus both the isoclines intersect each other in the region: $M = \{(x,y) : 0 < x < \psi_2, 0 < y < \frac{r_0}{a_1}\}$ in the following two cases: (see Fig.4.1)

$$Case(i) : \psi_2 > \psi_1, \frac{r_0}{a_1} > y_c \tag{35}$$

$$Case(ii) : \psi_2 < \psi_1, \frac{r_0}{a_1} < y_c \tag{36}$$

This point of intersection will give x^*, y^*. For uniqueness of (x^*, y^*), we must have $\frac{dy}{dx} < 0$ for both the curves in the region M.

For curve (33),

$$\frac{dy}{dx} = \frac{-(1 + k_1g_1'(x))[r_0 - r_1g_2(x,y)] - r_1g_2'(x,y)[K(g_1(x)) - x]}{a_1[K(g_1(x)) + r_0cx]} < 0 \tag{37}$$

and for curve (34),

$$\frac{dy}{dx} = \frac{-\delta_0\beta_{22}g_1'(x) - \delta_2\beta_{22}g_4'(x,y) + c_3\delta_3 g_3'(x,y)}{a_2\beta_2\delta_3} < 0 \tag{38}$$

In case (i), the absolute value of $\frac{dy}{dx}$ given by (37) is less than the absolute value of $\frac{dy}{dx}$ given by (38). For the case (ii), just the opposite is the condition.

Knowing the values of T^*, U^*, z^*, V^* and W^* can be computed from the equations (26)-(30).

Lemma 4.1: The set $\Omega_1 = \{(x,y,z,T,U,V,W) : 0 \le x(t) \le k_0, 0 \le \beta_2 y(t) + z(t) + T(t) + U(t) + V(t) + W(t) \le \frac{Q_0}{\Phi_2}, x(t) + y(t) + z(t) + T(t) \ge \frac{Q_0}{\Phi_4}$ and $z(t) + T(t) + U(t) + V(t) + W(t) \ge \frac{Q_0}{\Phi_3}\}$ is a region of attraction for all solutions initiating in the interior of the positive region, where $\Phi_2 = min\{(d_1 - a_1\beta_1\beta_2 x_u), d_2, \delta_0, \delta_1, \delta_2, \delta_3\}$, $\Phi_3 = max\{d_2 + c_3 z_u, \delta_0, \delta_1, \delta_2, \delta_3 + z_u\beta_{22}\}$, $\Phi_4 = max\{a_1 y_u + r_1 U_u(\frac{k_0 - k_1 T_l - x_l}{k_0 - k_1 T_u + r_0 c x_l}), a_2 z_u + d_1 + b_1 y_u + \beta_{11} V_u, d_2 + c_3 z_u + \beta_{22} W_u, \delta_0 + \alpha_1 x_u\}$.

Proof: From (1) we get,

$$\frac{dx}{dt} \le r_0 x(\frac{k_0 - x}{k_0 + r_0 c x})$$

then by the usual comparison theorem, we get as $t \to \infty$, $x \le k_0$.

Now, let us consider the following function: $w_2(t) = \beta_2 y(t) + z(t) + T(t) + U(t) + V(t) + W(t)$

by using (2) to (7), we get

$$\frac{dw_2}{dt} + \Phi_2 w_2 \le Q_0$$

where $\Phi_2 = min\{(d_1 - a_1\beta_1\beta_2 x_u), d_2, \delta_0, \delta_1, \delta_2, \delta_3\}$ and given that $d_1 > a_1\beta_1\beta_2 x_u$ then by the usual comparison theorem, we get as $t \to \infty$,

$$w_2(t) \le \frac{Q_0}{\Phi_2}$$

Again, let us consider the following function: $w_3(t) = z(t) + T(t) + U(t) + V(t) + W(t)$

by using (3) to (7), we get

$$\frac{dw_3}{dt} + \Phi_3 w_3 \ge Q_0$$

where $\Phi_3 = max\{d_2 + c_3 z_u, \delta_0, \delta_1, \delta_2, \delta_3 + z_u\beta_{22}\}$ then by the usual comparison theorem, we get as $t \to \infty$,

$$w_3(t) \ge \frac{Q_0}{\Phi_3}$$

Again, $w_4(t) = x(t) + y(t) + z(t) + T(t)$

by using (1) to (4), we get

$$\frac{dw_4}{dt} + \Phi_4 w_4 \ge Q_0$$

where $\Phi_4 = max\{a_1 y_u + r_1 U_u(\frac{k_0 - k_1 T_l - x_l}{k_0 - k_1 T_u + r_0 c x_l}), a_2 z_u + d_1 + b_1 y_u + \beta_{11} V_u, d_2 + c_3 z_u + \beta_{22} W_u, \delta_0 + \alpha_1 x_u\}$ then by the usual comparison theorem, we get as $t \to \infty$,

$$w_4(t) \ge \frac{Q_0}{\Phi_4}$$

This proves the lemma.

4.2. Dynamical behaviour of the Main Model

The stability behavior of $\hat{E_{10}}$ and $\check{E_{11}}$ can be studied by computing variational matrices, and $\bar{E_{12}}$ and E_{13}^* can be studied by computing Lyapunov's direct method.

The general variational matrix corresponding to the Main Model is

$$J(x,y,z,T,U,V,W) = \begin{bmatrix} -n_{11} & -n_{12} & 0 & -n_{14} & -n_{15} & 0 & 0 \\ n_{21} & n_{22} & -n_{23} & 0 & 0 & -n_{26} & 0 \\ 0 & n_{32} & n_{33} & 0 & 0 & 0 & -n_{37} \\ -n_{41} & 0 & 0 & -n_{44} & 0 & 0 & 0 \\ n_{51} & -n_{52} & 0 & n_{54} & -n_{55} & 0 & 0 \\ n_{61} & n_{62} & -n_{63} & 0 & n_{65} & -n_{66} & 0 \\ 0 & n_{72} & n_{73} & 0 & 0 & n_{76} & -n_{77} \end{bmatrix}$$

where,
$n_{11} = \mu_1 - r(U)\mu_2 + a_1 y$, $n_{12} = a_1 x$, $n_{14} = r(U)\mu_3$, $n_{15} = x r_1 \mu_2$, $n_{21} = a_1\beta_1 y$, $n_{22} = a_1\beta_1 x - a_2 z - \beta_{11} V - d_1 - 2b_1 y$, $n_{23} = a_2 y$ $n_{26} = \beta_{11} y$, $n_{32} = a_2\beta_2 z$, $n_{33} = a_2\beta_2 y - \beta_{22} W - d_2 - 2c_3 z$, $n_{37} = \beta_{22} z$, $n_{41} = \alpha_1 T$, $n_{44} = (\delta_0 + \alpha_1 x)$, $n_{51} = \alpha_1 T - a_1 a_3 U y$ $n_{52} = a_1 a_3 U x$, $n_{54} = \alpha_1 x$, $n_{55} = \delta_1 + a_1 a_3 x y$, $n_{61} = a_1 a_3 U y$, $n_{62} = a_1 a_3 U x - a_2 a_4 V z$, $n_{63} = a_2 a_4 V y$, $n_{65} = a_1 a_3 x y$ $n_{66} = (\delta_2 + a_2 a_4 y z)$, $n_{72} = a_2 a_4 V z$, $n_{73} = a_2 a_4 V y$, $n_{76} = a_2 a_4 y z$, $n_{77} = \delta_3$,

$$\mu_1 = \left(\frac{2xr(U)K(T)}{(K(T)+r_0cx)^2}\right),\ \mu_2 = \left(\frac{K(T)-x}{K(T)+r_0cx}\right),\ \mu_3 = \left(\frac{x^2k_1(1+rc)}{(K(T)+r_0cx)^2}\right).$$

- About $\hat{E_{10}}$, the eigenvalues of the characteristic equation are r_0, $-d_1$, $-d_2$, $-\delta_0$, $-\delta_1$, $-\delta_2$ and $-\delta_3$, which shows that $\hat{E_{10}}$ is unstable.
- About $\check{E_{11}}$, the eigenvalues of the characteristic equation are $-\mu_1$, $a_1\beta_1\check{x} - d_1$, $-d_2$, $-(\delta_0 + \alpha_1\check{x})$, $-\delta_1$, $-\delta_2$ and $-\delta_3$ which shows that $\check{E_{11}}$ is locally asymptotically stable if $k_1\check{T} < k_0$, $r_1\check{U} < r_0$, $a_1\beta_1\check{x} < d_1$ and $k_1 Q_0 < k_0\delta_0$ hold good.
  Remark 4: From the stability conditions of $\check{E_{11}}$ it may be noted that if (i) the carrying capacity of prey is positive, (ii) the growth rate of prey is positive, (iii) the product of the predation rate of intermediate predator, its conversion efficiency and the equilibrium of prey is less than the death rate of intermediate predator, and (iv) the rate of decrease of carrying capacity multiplied with the rate of introduction of toxicant into the environment is less than the product of carrying capacity and the first order decay rate of toxicant in the environment are satisfied then only prey population will survive.

Theorem 4.1: If the following inequalities hold

$$8(R_2 + \alpha_1\bar{T}N_3)^2 < R_1N_3(\delta_0 + \alpha_1\bar{x}) \tag{39}$$

$$16(R_3 - (\alpha_1\bar{T} - a_1a_3\bar{y}\bar{U})N_4)^2 < R_1N_4(\delta_1 + a_1a_3\bar{x}\bar{y}) \tag{40}$$

$$16(\beta_{11}\bar{y}N_1 - a_1a_3\bar{x}\bar{U}N_5)^2 < b_1\delta_2\bar{y}N_1N_5 \tag{41}$$

$$12N_5(a_2a_4\bar{V}\bar{y})^2 < \delta_2N_2(a_2\bar{y}\beta_2 - d_2) \tag{42}$$

$$16N_5(a_1a_3\bar{x}\bar{y})^2 < \delta_2N_4(\delta_1 + a_1a_3\bar{x}\bar{y}) \tag{43}$$

$$d_2 < a_2\beta_2\bar{y} \tag{44}$$

where,

$$N_1 = \frac{a_2\bar{x}}{a_1\beta_1\bar{y}} \tag{45}$$

$$N_2 > \frac{12N_1(a_2\bar{y})^2}{b_1\bar{y}(a_2\bar{y}\beta_2 - d_2)} \tag{46}$$

$$N_3 > \frac{8N_4(\alpha_1\bar{x})^2}{(\delta_0 + \alpha_1\bar{x})(\delta_1 + a_1a_3\bar{x}\bar{y})} \tag{47}$$

$$N_4 < \frac{b_1\bar{y}N_1(\delta_1 + a_1a_3\bar{x}\bar{y})}{(4a_1a_3\bar{x}\bar{U})^2} \tag{48}$$

$$N_5 < \frac{R_1\delta_2}{(4a_1a_3\bar{y}\bar{U})^2} \tag{49}$$

$$N_6 < \frac{N_2\delta_3(a_2\bar{y}\beta_2 - d_2)}{3(a_2a_4\bar{V}\bar{y})^2} \tag{50}$$

$$R_1 = \frac{\bar{x}(1 + r_0c)r(\bar{U})K(\bar{T})}{(K(\bar{T}) + r_0c\bar{x})^2}, R_2 = \frac{\bar{x}^2k_1(1 + r_0c)r(\bar{U})}{(K(\bar{T}) + r_0c\bar{x})^2}, R_3 = \frac{r_1\bar{x}(K(\bar{T}) - \bar{x})}{K(\bar{T}) + r_0c\bar{x}},$$

then the positive equilibrium $\bar{E_{12}}$ is locally asymptotically stable.
Proof: We first linearize the system about the equilibrium $\bar{E_{12}}$ by using the following transformations

$$x = \bar{x} + n_1; \quad y = \bar{y} + n_2; \quad z = \bar{z} + n_3; \quad T = \bar{T} + n_4;$$
$$U = \bar{U} + n_5; \quad V = \bar{V} + n_6; \quad W = \bar{W} + n_7;$$

where n_1, n_2, n_3, n_4, n_5, n_6 and n_7 are small perturbations around $\bar{E_{12}}$. Then we get the following linearized the system,

$$\begin{aligned}
\frac{dn_1}{dt} &= -R_1 n_1 - a_2\bar{x}n_2 - R_2 n_4 - R_3 n_5\\
\frac{dn_2}{dt} &= a_1\beta_1\bar{y}n_1 - b_1\bar{y}n_2 - a_2\bar{y}n_3 - \beta_{11}\bar{y}n_6\\
\frac{dn_3}{dt} &= -n_3(a_2\bar{y}\beta_2 - d_2)\\
\frac{dn_4}{dt} &= -\alpha_1\bar{T}n_1 - n_4(\delta_0 + \alpha_1\bar{x})\\
\frac{dn_5}{dt} &= n_1(\alpha_1\bar{T} - a_1a_3\bar{y}\bar{U}) - n_2a_1a_3\bar{U}\bar{x} + \alpha_1\bar{x}n_4 - n_5(\delta_1 + a_1a_3\bar{x}\bar{y})\\
\frac{dn_6}{dt} &= a_1a_3\bar{y}\bar{U}n_1 + a_1a_3\bar{x}\bar{U}n_2 - a_2a_4\bar{V}\bar{y}n_3 + a_1a_3\bar{y}\bar{x}n_5 - \delta_2 n_6\\
\frac{dn_7}{dt} &= a_2a_4\bar{V}\bar{y}n_3 - \delta_3 n_7
\end{aligned}$$

where,

$$R_1 = \frac{\bar{x}(1 + r_0c)r(\bar{U})K(\bar{T})}{(K(\bar{T}) + r_0c\bar{x})^2}, R_2 = \frac{\bar{x}^2k_1(1 + r_0c)r(\bar{U})}{(K(\bar{T}) + r_0c\bar{x})^2}, R_3 = \frac{r_1\bar{x}(K(\bar{T}) - \bar{x})}{K(\bar{T}) + r_0c\bar{x}}.$$

Now consider the following positive definite function

$$V_{11} = \frac{1}{2}n_1^2 + N_1\frac{1}{2}n_2^2 + N_2\frac{1}{2}n_3^2 + N_3\frac{1}{2}n_4^2 + N_4\frac{1}{2}n_5^2 + N_5\frac{1}{2}n_6^2 + N_6\frac{1}{2}n_7^2$$

$$\frac{dV_{11}}{dt} = n_1\frac{dn_1}{dt} + N_1n_2\frac{dn_2}{dt} + N_2n_3\frac{dn_3}{dt} + N_3n_4\frac{dn_4}{dt} + N_4n_5\frac{dn_5}{dt} + N_5n_6\frac{dn_6}{dt} + N_6n_7\frac{dn_7}{dt}$$

$$\begin{aligned}
\frac{dV_{11}}{dt} = &-R_1n_1^2 - n_2^2b_1\bar{y}N_1 - n_3^2N_2(a_2\bar{y}\beta_2 - d_2) - n_4^2N_3(\delta_0 + \alpha_1\bar{x})\\
&-n_5^2N_4(\delta_1 + a_1a_3\bar{x}\bar{y}) - n_6^2\delta_2N_5 - n_7^2\delta_3N_6 - n_1n_2(a_2\bar{x} - a_1\bar{y}N_1\beta_1)\\
&-n_1n_4(R_2 + \alpha_1\bar{T}N_3) - n_1n_5[R_3 - (\alpha_1\bar{T} - a_1a_3\bar{y}\bar{U})N_4] + n_1n_6a_1a_3\bar{y}\bar{U}N_5\\
&-n_2n_3a_2\bar{y}N_1 - n_2n_5a_1a_3\bar{U}\bar{x}N_4 - n_2n_6(\beta_{11}\bar{y}N_1 - a_1a_3\bar{U}\bar{x}N_5)\\
&-n_3n_6a_2a_4\bar{V}\bar{y}N_5 + n_3n_7a_2a_4\bar{V}\bar{y}N_6 + n_4n_5\bar{x}\alpha_1N_4 + n_5n_6a_1a_3\bar{x}\bar{y}N_5
\end{aligned}$$

Now using the sylvester's criterion in the quadratic forms

$$\begin{aligned}
\frac{dV_{11}}{dt} \leq &-[((b_{11}/2)n_1^2 + b_{12}n_1n_2 + (b_{22}/2)n_2^2) + ((b_{11}/2)n_1^2 + b_{14}n_1n_4 + (b_{44}/2)n_4^2)\\
&+((b_{11}/2)n_1^2 + b_{15}n_1n_5 + (b_{55}/2)n_5^2) + ((b_{11}/2)n_1^2 - b_{16}n_1n_6 + (b_{66}/2)n_6^2)\\
&+((b_{22}/2)n_2^2 + b_{23}n_2n_3 + (b_{33}/2)n_3^2) + ((b_{22}/2)n_2^2 + b_{25}n_2n_5 + (b_{55}/2)n_5^2)\\
&+((b_{22}/2)n_2^2 + b_{26}n_2n_6 + (b_{66}/2)n_6^2) + ((b_{33}/2)n_3^2 + b_{36}n_3n_6 + (b_{66}/2)n_6^2)\\
&+((b_{33}/2)n_3^2 - b_{37}n_3n_7 + (b_{77}/2)n_7^2) + ((b_{44}/2)n_4^2 - b_{45}n_4n_5 + (b_{55}/2)n_5^2)\\
&+((b_{55}/2)n_5^2 - b_{56}n_5n_6 + (b_{66}/2)n_6^2)]
\end{aligned}$$

where,
$b_{11} = R_1/4, b_{12} = a_2\bar{x} - a_1\bar{y}N_1\beta_1, b_{14} = R_2 + \alpha_1\bar{T}N_3, b_{15} = R_3 - (\alpha_1\bar{T} - a_1a_3\bar{y}\bar{U})N_4, b_{16} = a_1a_3\bar{y}\bar{U}N_5,$
$b_{22} = b_1\bar{y}N_1/4, b_{23} = a_2\bar{y}N_1, b_{25} = a_1a_3\bar{U}\bar{x}N_4, b_{26} = \beta_{11}\bar{y}N_1 - a_1a_3\bar{U}\bar{x}N_5, b_{33} = N_2(a_2\bar{y}\beta_2 - d_2)/3, b_{36} = a_2a_4\bar{V}\bar{y}N_5,$

$b_{37} = a_2a_4\bar{V}\bar{y}N_6, b_{44} = N_3(\delta_0 + \alpha_1\bar{x})/2, b_{45} = \bar{x}\alpha_1N_4, b_{55} = N_4(\delta_1 + a_1a_3\bar{x}\bar{y})/4, b_{56} = a_1a_3\bar{x}\bar{y}N_5, b_{66} = \delta_2N_5/4,$ $b_{77} = \delta_3N_6$. Sufficient conditions for dV_{11}/dt to be negative definite are that the following inequalities hold:

$$b_{33} > 0 \tag{51}$$
$$b_{11}b_{22} > b_{12}^2 \tag{52}$$
$$b_{11}b_{44} > b_{14}^2 \tag{53}$$
$$b_{11}b_{55} > b_{15}^2 \tag{54}$$
$$b_{11}b_{66} > b_{16}^2 \tag{55}$$
$$b_{22}b_{33} > b_{23}^2 \tag{56}$$
$$b_{22}b_{55} > b_{25}^2 \tag{57}$$
$$b_{22}b_{66} > b_{26}^2 \tag{58}$$
$$b_{33}b_{66} > b_{36}^2 \tag{59}$$
$$b_{33}b_{77} > b_{37}^2 \tag{60}$$
$$b_{44}b_{55} > b_{45}^2 \tag{61}$$
$$b_{55}b_{66} > b_{56}^2 \tag{62}$$

We note that the first, fourth, fifth, sixth, ninth and tenth inequalities, *i.e.*, $b_{11}b_{22} > b_{12}^2$, $b_{11}b_{66} > b_{16}^2$, $b_{22}b_{33} > b_{23}^2$, $b_{22}b_{55} > b_{25}^2$, $b_{33}b_{77} > b_{37}^2$ and $b_{44}b_{55} > b_{45}^2$ are satisfied due to the proper choice of N_1, N_2, N_3, N_4, N_5 and N_6 and other inequalities, (39) ⇒ (53), (40) ⇒ (54), (41) ⇒ (58), (42) ⇒ (59), (43) ⇒ (62) and (44) ⇒ (51). Hence V_{11} is a Lyapunov function with respect to $\bar{E_{12}}$, proving the theorem.

Theorem 4.2: If the following inequalities hold

$$8(R_{22} + \alpha_1T^*Q_3)^2 < R_{11}Q_3(\delta_0 + \alpha_1x^*) \tag{63}$$

$$20(R_{33} - (\alpha_1T^* - a_1a_3y^*U^*)Q_4)^2 < R_{11}Q_4(\delta_1 + a_1a_3x^*y^*) \tag{64}$$

$$25(a_2Q_1 - \beta_2z^*Q_2)^2 < b_1c_3y^*z^*Q_1Q_2 \tag{65}$$

$$20Q_4(a_1a_3x^*U^*)^2 < b_1y^*Q_1(\delta_1 + a_1a_3x^*y^*) \tag{66}$$

$$25(\beta_{11}y^*Q_1 - (a_1a_3x^*U^* - a_2a_4z^*V^*)Q_5)^2 < b_1y^*Q_1Q_5(\delta_2 + a_2a_4y^*z^*) \tag{67}$$

$$15Q_5(a_2a_4y^*V^*)^2 < c_3z^*Q_2(\delta_2 + a_2a_4y^*z^*) \tag{68}$$

$$15Q_6(a_2a_4y^*z^*)^2 < \delta_3Q_5(\delta_2 + a_2a_4y^*z^*) \tag{69}$$

where,

$$Q_1 = \frac{a_2x^*}{a_1\beta_1y^*} \tag{70}$$

$$Q_2 = \frac{a_2a_4y^*V^*Q_6}{\beta_{22}z^*} \tag{71}$$

$$Q_3 > \frac{8Q_4(\alpha_1x^*)^2}{(\delta_0 + \alpha_1x^*)(\delta_1 + a_1a_3x^*y^*)} \tag{72}$$

$$Q_4 > \frac{20Q_5(a_1a_3x^*y^*)^2}{(\delta_1 + a_1a_3x^*y^*)(\delta_2 + a_2a_4y^*z^*)} \tag{73}$$

$$Q_5 < \frac{R_{11}(\delta_2 + a_2a_4y^*z^*)}{20(a_1a_3y^*U^*)^2} \tag{74}$$

$$Q_6 < \frac{b_1\delta_3 Q_1 y^*}{15(a_2a_4V^*y^*)^2} \tag{75}$$

$$R_{11} = \frac{x^*(1+r_0c)r(U^*)K(T^*)}{(K(T^*)+r_0cx^*)^2}, R_{22} = \frac{{x^*}^2k_1(1+r_0c)r(U^*)}{(K(T^*)+r_0cx^*)^2}, R_{33} = \frac{r_1x^*(K(T^*)-x^*)}{K(T^*)+r_0cx^*},$$

then the positive equilibrium E^*_{13} is locally asymptotically stable.

Proof: We first linearize the system about the equilibrium E^*_{13} by using the following transformations

$$\begin{array}{llll} x = x^* + n_1; & y = y^* + n_2; & z = z^* + n_3; & T = T^* + n_4; \\ U = U^* + n_5; & V = V^* + n_6; & W = W^* + n_7; & \end{array}$$

where n_1, n_2, n_3, n_4, n_5, n_6 and n_7 are small perturbations around E^*_{13}. Then we get the following linearized the system,

$$\begin{aligned}
\frac{dn_1}{dt} &= -R_{11}n_1 - a_2x^*n_2 - R_{22}n_4 - R_{33}n_5 \\
\frac{dn_2}{dt} &= a_1\beta_1y^*n_1 - b_1y^*n_2 - a_2y^*n_3 - \beta_{11}y^*n_6 \\
\frac{dn_3}{dt} &= a_2\beta_2z^*n_2 - c_3z^*n_3 - \beta_{22}z^*n_7 \\
\frac{dn_4}{dt} &= -\alpha_1T^*n_1 - n_4(\delta_0 + \alpha_1x^*) \\
\frac{dn_5}{dt} &= n_1(\alpha_1T^* - a_1a_3y^*U^*) - n_2a_1a_3U^*x^* + \alpha_1x^*n_4 - n_5(\delta_1 + a_1a_3x^*y^*) \\
\frac{dn_6}{dt} &= a_1a_3y^*U^*n_1 + n_2(a_1a_3x^*U^* - a_2a_4z^*V^*) - a_2a_4V^*y^*n_3 + a_1a_3y^*x^*n_5 \\
&- n_6(\delta_2 + a_2a_4y^*z^*) \\
\frac{dn_7}{dt} &= a_2a_4V^*z^*n_2 + a_2a_4V^*y^*n_3 + a_2a_4z^*y^*n_6 - \delta_3n_7
\end{aligned}$$

where

$$R_{11} = \frac{x^*(1+r_0c)r(U^*)K(T^*)}{(K(T^*)+r_0cx^*)^2}, R_{22} = \frac{{x^*}^2k_1(1+r_0c)r(U^*)}{(K(T^*)+r_0cx^*)^2}, R_{33} = \frac{r_1x^*(K(T^*)-x^*)}{K(T^*)+r_0cx^*}.$$

Now consider the following positive definite function

$$V_{12} = \frac{1}{2}n_1^2 + Q_1\frac{1}{2}n_2^2 + Q_2\frac{1}{2}n_3^2 + Q_3\frac{1}{2}n_4^2 + Q_4\frac{1}{2}n_5^2 + +Q_5\frac{1}{2}n_6^2 + Q_6\frac{1}{2}n_7^2$$

$$\frac{dV_{12}}{dt} = n_1\frac{dn_1}{dt} + Q_1n_2\frac{dn_2}{dt} + Q_2n_3\frac{dn_3}{dt} + Q_3n_4\frac{dn_4}{dt} + Q_4n_5\frac{dn_5}{dt} + Q_5n_6\frac{dn_6}{dt} + Q_6n_7\frac{dn_7}{dt}$$

$$\begin{aligned}
\frac{dV_{12}}{dt} = \; & -R_{11}n_1^2 - n_2^2b_1y^*Q_1 - n_3^2c_3z^*Q_2 - n_4^2Q_3(\delta_0 + \alpha_1x^*) - n_5^2Q_4(\delta_1 + a_1a_3x^*y^*) \\
& -n_6^2(\delta_2 + a_2a_4y^*z^*)Q_5 - n_7^2\delta_3Q_6 - n_1n_2(a_2x^* - a_1y^*Q_1\beta_1) \\
& -n_1n_4(R_{22} + \alpha_1T^*Q_3) - n_1n_5[R_{33} - (\alpha_1T^* - a_1a_3y^*U^*)Q_4] \\
& +n_1n_6a_1a_3y^*U^*Q_5 - n_2n_3a_2(y^*Q_1 - \beta_2z^*Q_2) - n_2n_5a_1a_3U^*x^*Q_4 \\
& -n_2n_6(\beta_{11}y^*Q_1 - (a_1a_3U^*x^* - a_2a_4V^*z^*)Q_5) + n_2n_7a_2a_4V^*z^*Q_6 \\
& -n_3n_6a_2a_4V^*y^*Q_5 - n_3n_7(\beta_{22}z^*Q_2 - a_2a_4V^*y^*)Q_6 + n_4n_5x^*\alpha_1Q_4 \\
& +n_5n_6a_1a_3x^*y^*Q_5 + n_6n_7a_2a_4y^*z^*Q_6
\end{aligned}$$

Now using the sylvester's criterion in the quadratic forms

$$\begin{aligned}\frac{dV_{12}}{dt} \leq\ & -[((e_{11}/2)n_1^2 + e_{12}n_1n_2 + (e_{22}/2)n_2^2) + ((e_{11}/2)n_1^2 + e_{14}n_1n_4 + (e_{44}/2)n_4^2)\\ & +((e_{11}/2)n_1^2 + e_{15}n_1n_5 + (e_{55}/2)n_5^2) + ((e_{11}/2)n_1^2 - e_{16}n_1n_6 + (e_{66}/2)n_6^2)\\ & +((e_{22}/2)n_2^2 + e_{23}n_2n_3 + (e_{33}/2)n_3^2) + ((e_{22}/2)n_2^2 + e_{25}n_2n_5 + (e_{55}/2)n_5^2)\\ & +((e_{22}/2)n_2^2 + e_{26}n_2n_6 + (e_{66}/2)n_6^2) + ((e_{22}/2)n_2^2 - e_{27}n_2n_7 + (e_{77}/2)n_7^2)\\ & +((e_{33}/2)n_3^2 + e_{36}n_3n_6 + (e_{66}/2)n_6^2) + ((e_{33}/2)n_3^2 + e_{37}n_3n_7 + (e_{77}/2)n_7^2)\\ & +((e_{44}/2)n_4^2 - e_{45}n_4n_5 + (e_{55}/2)n_5^2) + ((e_{55}/2)n_5^2 - e_{56}n_5n_6 + (e_{66}/2)n_6^2)\\ & +((e_{66}/2)n_6^2 - e_{67}n_6n_7 + (e_{77}/2)n_7^2)]\end{aligned}$$

where,
$e_{11} = R_{11}/4, e_{12} = a_2x^* - a_1y^*Q_1\beta_1, e_{14} = R_{22} + \alpha_1T^*Q_3, e_{15} = R_{33} - (\alpha_1T^* - a_1a_3y^*U^*)Q_4, e_{16} = a_1a_3y^*U^*Q_5,$
$e_{22} = b_1y^*Q_1/5, e_{23} = a_2(y^*Q_1 - \beta_2z^*Q_2), e_{25} = a_1a_3U^*x^*Q_4, e_{26} = \beta_{11}y^*Q_1 - (a_1a_3U^*x^* - a_2a_4V^*z^*)Q_5,$
$e_{27} = a_2a_4V^*z^*Q_6, e_{33} = c_3z^*Q_2/3, e_{36} = a_2a_4V^*y^*Q_5, e_{37} = (\beta_{22}z^*Q_2 - a_2a_4V^*y^*)Q_6, e_{44} = Q_3(\delta_0 + \alpha_1x^*)/2,$
$e_{45} = x^*\alpha_1Q_4, e_{55} = Q_4(\delta_1 + a_1a_3x^*y^*)/4, e_{56} = a_1a_3x^*y^*Q_5, e_{66} = (\delta_2 + a_2a_4y^*z^*)Q_5/5, e_{67} = a_2a_4y^*z^*Q_6,$
$e_{77} = \delta_3Q_6/3$. Sufficient conditions for dV_{12}/dt to be negative definite are that the following inequalities hold:

$$e_{11}e_{22} > e_{12}^2 \tag{76}$$
$$e_{11}e_{44} > e_{14}^2 \tag{77}$$
$$e_{11}e_{55} > e_{15}^2 \tag{78}$$
$$e_{11}e_{66} > e_{16}^2 \tag{79}$$
$$e_{22}e_{33} > e_{23}^2 \tag{80}$$
$$e_{22}e_{55} > e_{25}^2 \tag{81}$$
$$e_{22}e_{66} > e_{26}^2 \tag{82}$$
$$e_{22}e_{77} > e_{27}^2 \tag{83}$$
$$e_{33}e_{66} > e_{36}^2 \tag{84}$$
$$e_{33}e_{77} > e_{37}^2 \tag{85}$$
$$e_{44}e_{55} > e_{45}^2 \tag{86}$$
$$e_{55}e_{66} > e_{56}^2 \tag{87}$$
$$e_{66}e_{77} > e_{67}^2 \tag{88}$$

We note that the first, fourth, eighth, tenth, eleventh and twelfth inequalities, *i.e.*, $e_{11}e_{22} > e_{12}^2$, $e_{11}e_{66} > e_{16}^2$, $e_{22}e_{77} > e_{27}^2$, $e_{33}e_{77} > e_{37}^2$, $e_{44}e_{55} > e_{45}^2$ and $e_{55}e_{66} > e_{56}^2$ are satisfied due to the proper choice of Q_1, Q_2, Q_3, Q_4, Q_5 and Q_6 and other inequalities, (63) ⇒ (77), (64) ⇒ (78), (65) ⇒ (80), (66) ⇒ (81), (67) ⇒ (82), (68) ⇒ (84) and (69) ⇒ (88). Hence V_{12} is a Lyapunov function with respect to E_{13}^*, proving the theorem.

Theorem 4.3: If the following inequalities hold in the region Ω_1

$$8(\pi_3 + \alpha_1T_lL_3)^2 < \pi_1L_3(\delta_0 + \alpha_1x^*) \tag{89}$$

$$16(\pi_2 - L_4(\alpha_1T^* - a_1a_3U_ly^*))^2 < \pi_1L_4(\delta_1 + a_1a_3x^*y^*) \tag{90}$$

$$20L_4(a_1a_3x_lU_l)^2 < L_1(\delta_1 + a_1a_3x^*y^*)[d_1 + b_1(y_u + y^*) - a_1\beta_1x_l + a_2z_u + \beta_{11}V^*] \tag{91}$$

$$25[\beta_{11}y_lL_1 - L_5(a_1a_3U^*x_u - a_2a_4V_lz^*)]^2$$

$$< L_1L_5(\delta_2 + a_2a_4y^*z^*)[d_1 + b_1(y_u + y^*) - a_1\beta_1x_l + a_2z_u + \beta_{11}V^*] \tag{92}$$

$$15L_6(a_2a_4V_lz^*)^2 < \delta_2L_1[d_1 + b_1(y_u + y^*) - a_1\beta_1x_l + a_2z_u + \beta_{11}V^*] \tag{93}$$

$$15L_5(a_2a_4V_ly_l)^2 < L_2(\delta_2 + a_2a_4y^*z^*)[d_2 + c_3(z_u + z^*) - a_2\beta_2y_l + \beta_{22}W_u] \tag{94}$$

$$9(L_2\beta_{22}z^* - a_2a_4V_uy_uL_6)^2 < \delta_2L_2L_6[d_2 + c_3(z_u + z^*) - a_2\beta_2y_l + \beta_{22}W_u] \quad (95)$$

where

$$L_1 = \frac{1}{\beta_1y^*} \quad (96)$$

$$L_2 = \frac{a_2y^*L_1}{a_2z_u\beta_2} \quad (97)$$

$$L_3 > \frac{8L_4(\alpha_1x_l)^2}{(\delta_0 + \alpha_1x^*)(\delta_1 + a_1a_3x^*y^*)} \quad (98)$$

$$L_4 > \frac{20L_5(a_1a_3x_ly_l)^2}{(\delta_1 + a_1a_3x^*y^*)(\delta_2 + a_2a_4y^*z^*)} \quad (99)$$

$$L_5 < \frac{\pi_1(\delta_2 + a_2a_4y^*z^*)}{20(a_1a_3y^*U^*)^2} \quad (100)$$

$$L_6 < \frac{L_1\delta_2(\delta_2 + a_2a_4y^*z^*)}{15(a_2a_4y^*z^*)^2} \quad (101)$$

$\pi_1 = \frac{(1+r_0c)K(T^*)r(U^*)}{(K(T^*)+r_0cx^*)(K(T_l)+r_0cx_l)}$, $\pi_2 = \frac{K(T^*)(k_0+r_0cx^*-x^*)+k_1T^*(k_1T_l-k_0)-r_0cx_ux^*}{(K(T^*)+r_0cx^*)(K(T_u)+r_0cx_u)}$, $\pi_3 = \frac{k_1x^*r(U^*)+r_0ck_1x^*r(U_l)+r_1k_0k_1(U^*-U_u)}{(K(T^*)+r_0cx^*)(K(T_u)+r_0cx_u)}$, then the positive equilibrium E_{13}^* is globally asymptotically stable with respect to all solutions initiating in the interior of positive region Ω_1.

Proof: We consider the following positive definite function about E_{13}^*:

$$V_{13} = (x - x^* - x^*ln(\frac{x}{x^*})) + \frac{L_1}{2}(y-y^*)^2 + \frac{L_2}{2}(z-z^*)^2 + \frac{L_3}{2}(T-T^*)^2 + \frac{L_4}{2}(U-U^*)^2 + \frac{L_5}{2}(V-V^*)^2 + \frac{L_6}{2}(W-W^*)^2$$

Differentiating V_{13} with respect to time t, we get

$$\frac{dV_{13}}{dt} = (\frac{x-x^*}{x})\frac{dx}{dt} + L_1(y-y^*)\frac{dy}{dt} + L_2(z-z^*)\frac{dz}{dt} + L_3(T-T^*)\frac{dT}{dt} + L_4(U-U^*)\frac{dU}{dt} + L_5(V-V^*)\frac{dV}{dt} + L_6(W-W^*)\frac{dW}{dt}$$

$$\begin{aligned}\frac{dV_{13}}{dt} &= -(x-x^*)^2\pi_1 - (y-y^*)^2L_1[d_1 + b_1(y_u+y^*) - a_1\beta_1x_l + a_2z_u + \beta_{11}V^*] \\ &\quad -(z-z^*)^2L_2[d_2 + c_3(z_u+z^*) - a_2\beta_2y_l + \beta_{22}W_u] - (T-T^*)^2L_3(\delta_0 + \alpha_1x^*) \\ &\quad -(U-U^*)^2L_4(\delta_1 + a_1a_3x^*y^*) - (V-V^*)^2L_5(\delta_2 + a_2a_4y^*z^*) \\ &\quad -(W-W^*)^2L_6\delta_2 - (x-x^*)(y-y^*)a_1(1 - L_1\beta_1y^*) \\ &\quad -(x-x^*)(T-T^*)(\pi_3 + \alpha_1T_lL_3) + (x-x^*)(V-V^*)a_1a_3U^*y^*L_5 \\ &\quad -(x-x^*)(U-U^*)(\pi_2 - L_4(\alpha_1T^* - a_1a_3U_ly^*)) \\ &\quad -(y-y^*)(z-z^*)(a_2y^*L_1 - L_2a_2\beta_2z) - (y-y^*)(U-U^*)a_1a_3L_4Ux \\ &\quad -(y-y^*)(V-V^*)[\beta_{11}y_lL_1 - L_5(a_1a_3U^*x_u - a_2a_4V_lz^*)] \\ &\quad +(y-y^*)(W-W^*)a_2a_4Vz^*L_6 - (z-z^*)(V-V^*)a_2a_4yVL_5 \\ &\quad +(z-z^*)(W-W^*)(L_2\beta_{22}z^* - a_2a_4VyL_6) + (T-T^*)(U-U^*)x\alpha_1L_4 \\ &\quad +(U-U^*)(V-V^*)a_1a_3xyL_5 + (V-V^*)(W-W^*)a_2a_4y^*z^*L_6\end{aligned}$$

Now using the sylvester's criterion in the quadratic forms

$$\begin{aligned}\frac{dV_{13}}{dt} \leq & -[((f_{11}/2)(x-x^*)^2 + f_{12}(x-x^*)(y-y^*) + (f_{22}/2)(y-y^*)^2) \\ & +((f_{11}/2)(x-x^*)^2 + f_{14}(x-x^*)(T-T^*) + (f_{44}/2)(T-T^*)^2) \\ & +((f_{11}/2)(x-x^*)^2 + f_{15}(x-x^*)(U-U^*) + (f_{55}/2)(U-U^*)^2) \\ & +((f_{11}/2)(x-x^*)^2 - f_{16}(x-x^*)(V-V^*) + (f_{66}/2)(V-V^*)^2) \\ & +((f_{22}/2)(y-y^*)^2 + f_{23}(y-y^*)(z-z^*) + (f_{33}/2)(z-z^*)^2) \\ & +((f_{22}/2)(y-y^*)^2 + f_{25}(y-y^*)(U-U^*) + (f_{55}/2)(U-U^*)^2) \\ & +((f_{22}/2)(y-y^*)^2 + f_{26}(y-y^*)(V-V^*) + (f_{66}/2)(V-V^*)^2) \\ & +((f_{22}/2)(y-y^*)^2 - f_{27}(y-y^*)(W-W^*) + (f_{77}/2)(W-W^*)^2) \\ & +((f_{33}/2)(z-z^*)^2 + f_{36}(z-z^*)(V-V^*) + (f_{66}/2)(V-V^*)^2) \\ & +((f_{33}/2)(z-z^*)^2 + f_{37}(z-z^*)(W-W^*) + (f_{77}/2)(W-W^*)^2) \\ & +((f_{44}/2)(T-T^*)^2 - f_{45}(T-T^*)(U-U^*) + (f_{55}/2)(U-U^*)^2) \\ & +((f_{55}/2)(U-U^*)^2 - f_{56}(U-U^*)(V-V^*) + (f_{66}/2)(V-V^*)^2) \\ & +((f_{66}/2)(V-V^*)^2 - f_{67}(V-V^*)(W-W^*) + (f_{77}/2)(W-W^*)^2)]\end{aligned}$$

where,
$f_{11} = \pi_1/4, f_{12} = a_1(1 - L_1\beta_1 y^*), f_{14} = \pi_3 + \alpha_1 T L_3, f_{15} = \pi_2 - L_4(\alpha_1 T^* - a_1 a_3 U y^*), f_{16} = a_1 a_3 U^* y^* L_5,$
$f_{22} = L_1[d_1 + b_1(y + y^*) - a_1\beta_1 x + a_2 z + \beta_{11} V^*]/5, f_{23} = a_2 y^* L_1 - L_2 a_2 \beta_2 z, f_{25} = a_1 a_3 L_4 U x,$
$f_{26} = \beta_{11} y L_1 - L_5(a_1 a_3 U^* x - a_2 a_4 V z^*), f_{27} = a_2 a_4 V z^* L_6, f_{33} = L_2[d_2 + c_3(z + z^*) - a_2\beta_2 y + \beta_{22} W]/3,$
$f_{36} = a_2 a_4 V y L_5, f_{37} = L_2\beta_{22} z^* - a_2 a_4 V y L_6, f_{44} = (L_3/2)(\delta_0 + \alpha_1 x^*), f_{45} = x\alpha_1 L_4, f_{55} = (L_4/4)(\delta_1 + a_1 a_3 x^* y^*),$
$f_{56} = a_1 a_3 x y L_5, f_{66} = (L_5/5)(\delta_2 + a_2 a_4 y^* z^*), f_{67} = a_2 a_4 y^* z^* L_6, f_{77} = \delta_2 L_6/3$, $\pi_1 = \frac{1}{M_{11}}(1 + rc)K(T^*)r(U^*)$,
$\pi_2 = \frac{1}{M_{11}}(K(T^*)(k_0 + rcx^* - x^*) + k_1 T^*(k_1 T - k_0) - rcxx^*)$, $\pi_3 = \frac{1}{M_{11}}(k_1 x^* r(U^*) + rck_1 x^* r(U) + r_1 k_0 k_1(U^* - U))$,
$M_{11} = (K(T^*) + rcx^*)(K(T) + rcx)$
Sufficient conditions for dV_{13}/dt to be negative definite are that the following inequalities hold:

$$f_{11} f_{22} > f_{12}^2 \tag{102}$$
$$f_{11} f_{44} > f_{14}^2 \tag{103}$$
$$f_{11} f_{55} > f_{15}^2 \tag{104}$$
$$f_{11} f_{66} > f_{16}^2 \tag{105}$$
$$f_{22} f_{33} > f_{23}^2 \tag{106}$$
$$f_{22} f_{55} > f_{25}^2 \tag{107}$$
$$f_{22} f_{66} > f_{26}^2 \tag{108}$$
$$f_{22} f_{77} > f_{27}^2 \tag{109}$$
$$f_{33} f_{66} > f_{36}^2 \tag{110}$$
$$f_{33} f_{77} > f_{37}^2 \tag{111}$$
$$f_{44} f_{55} > f_{45}^2 \tag{112}$$
$$f_{55} f_{66} > f_{56}^2 \tag{113}$$
$$f_{66} f_{77} > f_{67}^2 \tag{114}$$

We note that the first, fourth, fifth, eleventh, twelfth and thirteenth inequalities, *i.e.*, $f_{11}f_{22} > f_{12}^2$, $f_{11}f_{66} > f_{16}^2$, $f_{22}f_{33} > f_{23}^2$, $f_{44}f_{55} > f_{45}^2$, $f_{55}f_{66} > f_{56}^2$ and $f_{66}f_{77} > f_{67}^2$ are satisfied due to the proper choice of L_1, L_2, L_3, L_4, L_5 and L_6 and other inequalities, (89) $\Rightarrow$ (103), (90) $\Rightarrow$ (104), (91) $\Rightarrow$ (107), (92) $\Rightarrow$ (108), (93) $\Rightarrow$ (109), (94) $\Rightarrow$ (110) and (95) $\Rightarrow$ (111). Hence V_{13} is a Lyapunov function with respect to E_{13}^*, whose domain contains the region of attraction Ω_1, proving the theorem.

5. Numerical Simulation

In this section, we demonstrate the dynamical behaviour of a three species food chain system with "food-limited" growth of prey population with and without toxicant with the help of numerical simulations to facilitate the interpretation of our mathematical findings. The figures illustrate the stability behaviour of all the equilibrium points of the models for the given sets of parameters and the graphs have been plotted with the help of MATLAB.

5.1. Numerical Simulation for Sub-Model

We choose the following values of parameters for $\check{E_{21}}$:

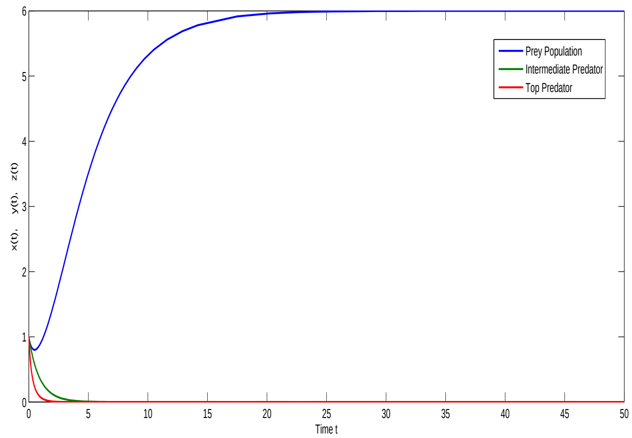

Fig.1: Time series graph for the Sub-Model around the equilibrium point $\check{E_{21}} = (\check{x}, 0, 0)$ showing the stability behavior.

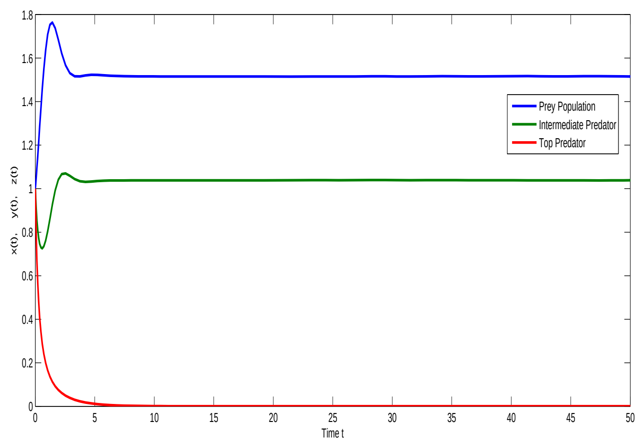

Fig.2: Time series graph for the Sub-Model around the equilibrium point $\bar{E_{22}} = (\bar{x}, \bar{y}, 0)$ showing the stability behavior.

$$r_0 = 5.8, \quad c = 4.0, \quad \beta_1 = 0.011, \quad a_1 = 2.11, \quad d_1 = 1.28, \quad c_3 = 1.9850,$$
$$k_0 = 6.0, \quad b_1 = 0.124, \quad \beta_2 = 0.018, \quad a_2 = 0.051, \quad d_2 = 2.0295.$$

It is found that under the above set of parameters, the equilibrium point $\check{E_{21}}$

$$\check{x} = 6.0, \quad \check{y} = 0.0, \quad \check{z} = 0.0$$

is locally asymptotically stable (see Fig.1).
We choose the following values of parameters for $\bar{E_{22}}$:

$$r_0 = 6.0, \quad b_1 = 1.124, \quad \beta_1 = 0.83, \quad a_1 = 1.4671, \quad d_1 = 0.678, \quad c_3 = 2.9830,$$
$$k_0 = 16.2, \quad c = 4.58, \quad \beta_2 = 0.18, \quad a_2 = 0.8870, \quad d_2 = 0.695.$$

It is found that under the above set of parameters, the equilibrium point $\bar{E_{22}}$

$$\bar{x} = 1.5156, \quad \bar{y} = 1.0385, \quad \bar{z} = 0.0$$

is locally asymptotically stable (see Fig.2).
We choose the following values of parameters for E^*_{23}:

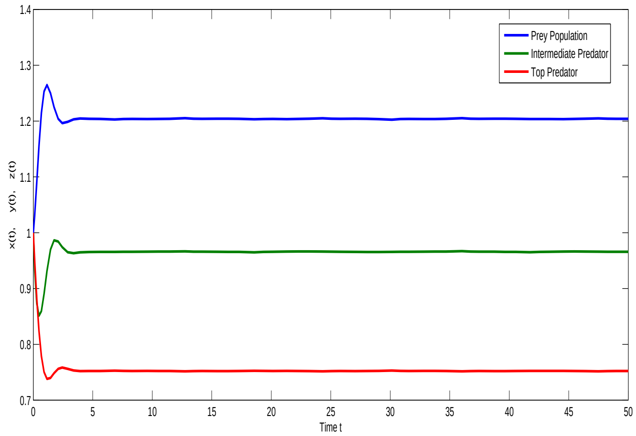

Fig.3: Time series graph for the Sub-Model around the equilibrium point $E^*_{23} = (x^*, y^*, z^*)$ showing the stability behavior.

$$r_0 = 6.8, \quad b_1 = 1.124, \quad \beta_1 = 1.02, \quad a_1 = 1.9672, \quad d_1 = 0.58, \quad c_3 = 1.9830,$$
$$k_0 = 16.21, \quad c = 4.58, \quad \beta_2 = 1.6, \quad a_2 = 0.9970, \quad d_2 = 0.049.$$

It is found that under the above set of parameters, the equilibrium point E^*_{23}

$$x^* = 1.2039, \quad y^* = 0.9657, \quad z^* = 0.7522$$

is locally asymptotically stable (see Fig.3).

5.2. Numerical Simulation for Main Model

We choose the following values of parameters for $\check{E_{11}}$:

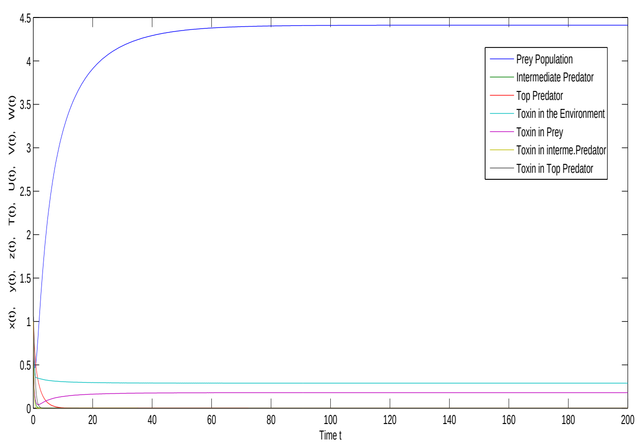

Fig.4: Time series graph for the Main Model around the equilibrium point $\check{E_{11}}(\check{x}, 0, 0, \check{T}, \check{U}, 0, 0)$ showing the stability behavior.

$$r_0 = 3.05, \quad c = 5.58, \quad \beta_1 = 0.2, \quad a_1 = 2.22, \quad \delta_0 = 5.52, \quad Q_0 = 1.988,$$
$$r_1 = 17.0, \quad c_3 = 0.02, \quad \beta_{11} = 1.1, \quad a_2 = 0.9813, \quad \delta_1 = 2.2, \quad d_1 = 2.45,$$
$$k_0 = 8.2, \quad \alpha_1 = 0.31, \quad \beta_2 = 1.6, \quad a_3 = 2.865, \quad \delta_2 = 5.890, \quad d_2 = 0.49,$$
$$k_1 = 6.0, \quad b_1 = 1.1231, \quad \beta_{22} = 1.15, \quad a_4 = 4.21, \quad \delta_3 = 2.13,$$

It is found that under the above set of parameters, the equilibrium point $\check{E_{11}}$

$$\check{x} = 4.4111, \quad \check{y} = 0, \quad \check{z} = 0, \quad \check{T} = 0.2890, \quad \check{U} = 0.1793, \quad \check{V} = 0, \quad \check{W} = 0$$

is locally asymptotically stable (see Fig.4).

We choose the following values of parameters for $\bar{E_{12}}$:

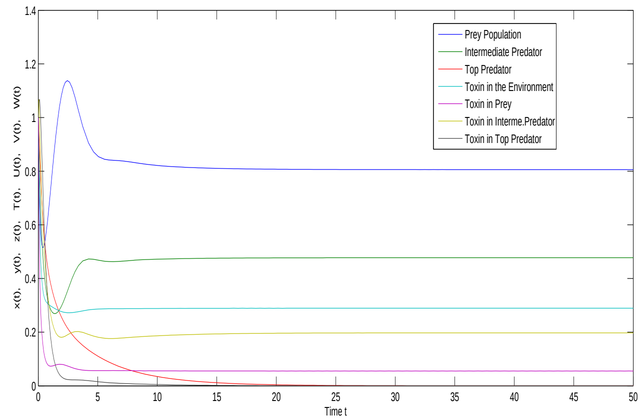

Fig.5: Time series graph for the Main Model around the equilibrium point $\bar{E_{12}}(\bar{x}, \bar{y}, 0, \bar{T}, \bar{U}, \bar{V}, 0)$ showing the stability behavior.

$$\begin{array}{llllll}
r_0 = 5.00, & c_3 = 0.02, & \beta_1 = 0.4, & a_1 = 3.22, & \delta_0 = 5.82, & Q_0 = 1.988, \\
r_1 = 11.0, & c = 5.58, & \beta_{11} = 1.01, & a_2 = 0.9913, & \delta_1 = 2.0, & d_1 = 0.35, \\
k_0 = 14.2, & \alpha_1 = 1.31, & \beta_2 = 0.6, & a_3 = 2.865, & \delta_2 = 1.9890, & d_2 = 0.5, \\
k_1 = 3.0, & b_1 = 1.0231, & \beta_{22} = 1.15, & a_4 = 4.21, & \delta_3 = 2.9, &
\end{array}$$

It is found that under the above set of parameters, the equilibrium point $\bar{E_{12}}$

$$\bar{x} = 0.8057, \quad \bar{y} = 0.4774, \quad \bar{z} = 0, \quad \bar{T} = 0.2892, \quad \bar{U} = 0.0549, \quad \bar{V} = 0.1974, \quad \bar{W} = 0$$

is locally asymptotically stable (see Fig.5).

We choose the following values of parameters for E_{13}^*:

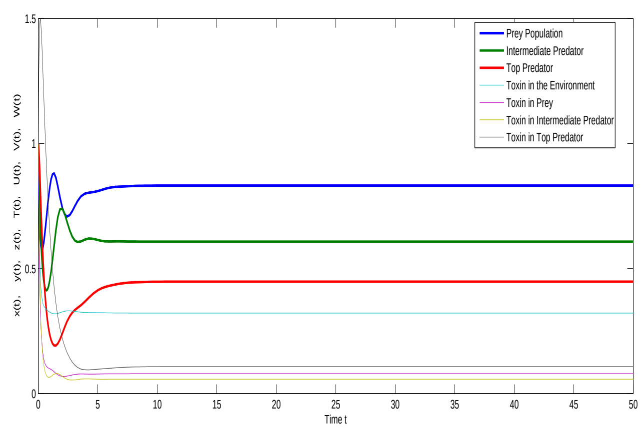

Fig.6: Time series graph for the Main Model around the equilibrium point $E_{13}^*(x^*, y^*, z^*, T^*, U^*, V^*, W^*)$ showing the stability behavior.

$$
\begin{array}{llllll}
r_0 = 5.66, & c_3 = 1.02, & \beta_1 = 1.2, & a_1 = 3.22, & \delta_0 = 7.52, & Q_0 = 2.988, \\
r_1 = 11.0, & c = 4.25, & \beta_{11} = 1.1, & a_2 = 2.13, & \delta_1 = 2.5, & d_1 = 1.45, \\
k_0 = 16.2, & \alpha_1 = 2.12, & \beta_2 = 1.6, & a_3 = 2.865, & \delta_2 = 3.99, & d_2 = 1.49, \\
k_1 = 3.0, & b_1 = 1.231, & \beta_{22} = 1.15, & a_4 = 4.21, & \delta_3 = 1.3, &
\end{array}
$$

It is found that under the above set of parameters, the equilibrium point E_{13}^*

$$
\begin{array}{llll}
x^* = 0.8320, & y^* = 0.6076, & z^* = 0.4477, & T^* = 0.3218, \\
U^* = 0.0793, & V^* = 0.0574, & W^* = 0.1079 &
\end{array}
$$

is locally asymptotically stable (see Fig.6).

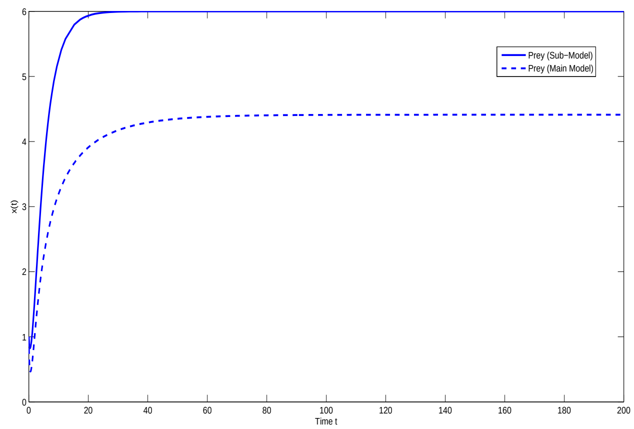

Fig.7: Time series graph of Prey population for Sub-Model and Main Model around the equilibrium points $\check{E_{21}} = (\check{x}, 0, 0)$ and $\check{E_{11}}(\check{x}, 0, 0, \check{T}, \check{U}, 0, 0)$.

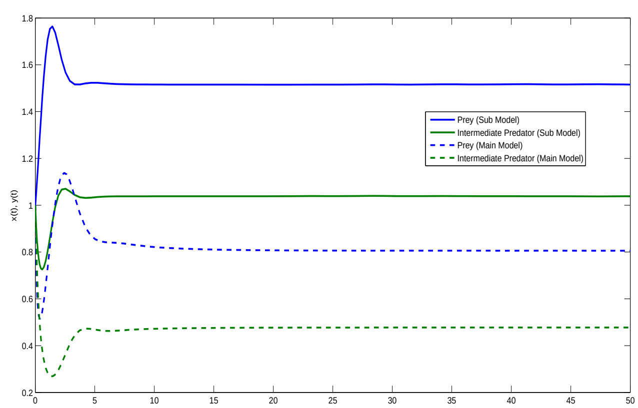

Fig.8: Time series graph of Prey and Intermediate Predator populations for Sub-Model and Main Model around the equilibrium points $\bar{E_{22}} = (\bar{x}, \bar{y}, 0)$ and $\bar{E_{12}}(\bar{x}, \bar{y}, 0, \bar{T}, \bar{U}, \bar{V}, 0)$.

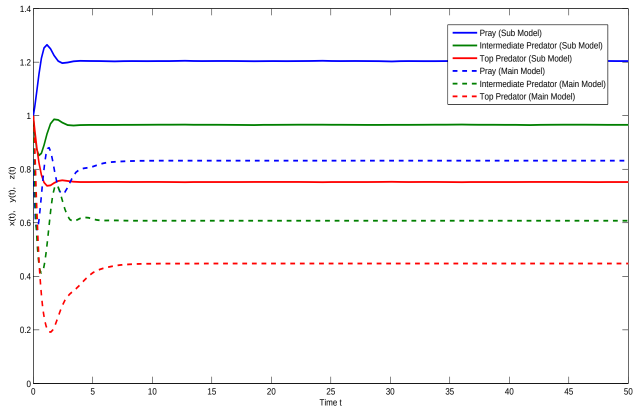

Fig.9: Time series graph of Prey, Intermediate Predator and Top Predator populations for Sub-Model and Main Model around the equilibrium points $E_{23}^* = (x^*, y^*, z^*)$ and $E_{13}^*(x^*, y^*, z^*, T^*, U^*, V^*, W^*)$.

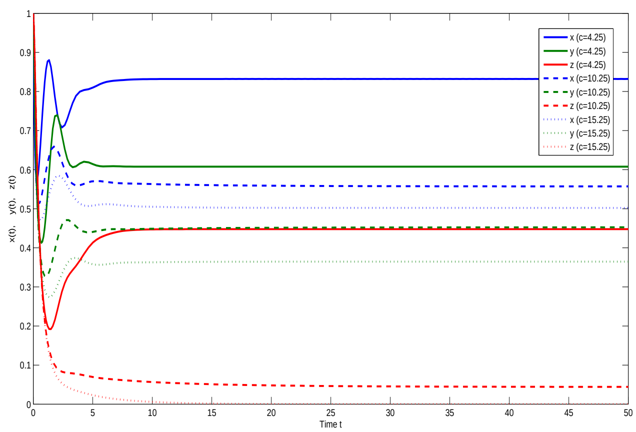

Fig.10: Variation of x, y and z with respect to time t, corresponding to different values of c in Main Model.

6. Conclusion

In this paper we have proposed and analyzed a nonlinear mathematical model for the effect of toxicant in a three species food-chain system with "food-limited" growth of prey population. It is concluded from the stability of $\check{E}_{11}$ of Main Model that only the prey population will survive and intermediate predator and top predator populations would tend to extinction. From the stability of $\check{E}_{21}$ of Sub-Model we derive the same dynamical behavior of prey and predator populations as observed for $\check{E}_{11}$ of Main Model with the only difference that equilibrium level of prey population is reduced due to the presence of toxicant (see Figs.1, 4 and 7). It is concluded from the stability of $\bar{E}_{12}$ of Main Model that the prey population and intermediate predator populations will survive and top predator population would tend to extinction. From the stability of $\bar{E}_{22}$ of Sub-Model we derive the same dynamical behavior of prey and predator populations as observed for $\bar{E}_{12}$ of Main Model with the only difference that equilibrium levels of prey and intermediate predator populations are reduced due to the presence of toxicant (see Figs.2, 5 and 8). The interior equilibrium points of both the models are locally and also globally stable showing the co-existence of all the three populations of prey and predator species. However, from the equilibrium values it is seen that the equilibrium density of top predator reduces due to the presence of toxicant in prey and intermediate predator (see Figs.3, 6 and 9). It may be also noted from the equilibrium of the intermediate predator population that the level of intermediate predator population may increase due to the decrease in the top predator density on account of toxicant (see Figs.9, 12, 14).

From Table 1, it may be observed that if we increase the toxicant input rate then both the predators may tend

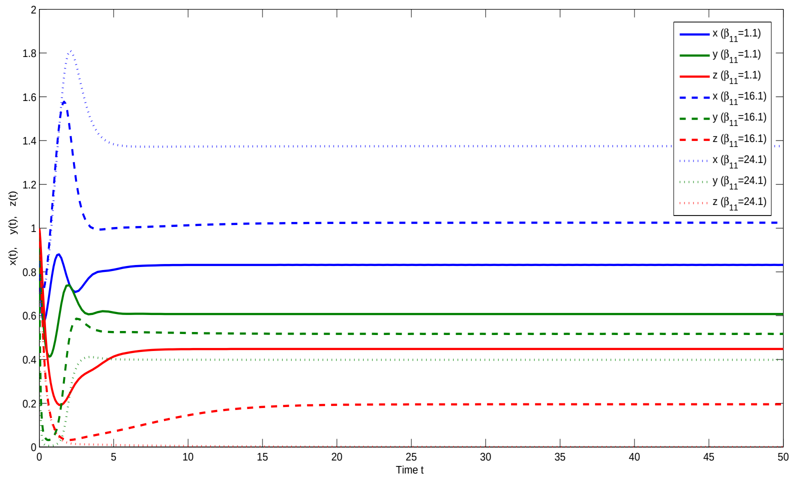

Fig.11: Variation of x, y and z with respect to time t, corresponding to different values of β_{11} in Main Model.

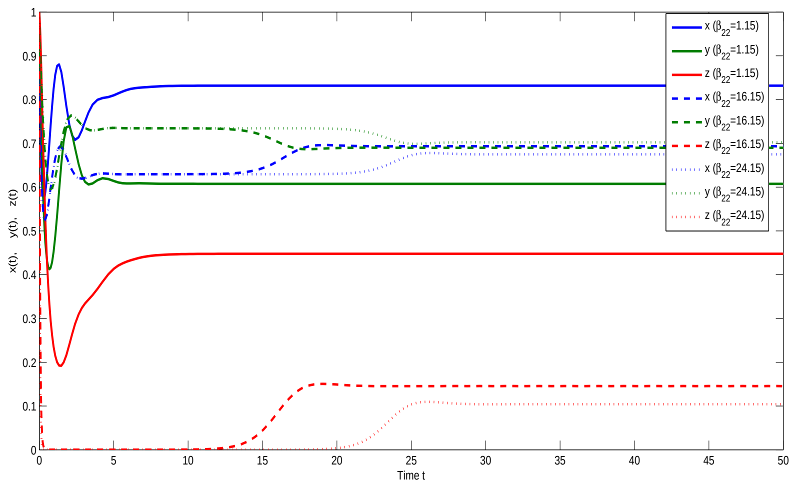

Fig.12: Variation of x, y and z with respect to time t, corresponding to different values of β_{22} in Main Model.

to extinction and if we decrease the toxicant input rate then the equilibria of all the three species will increase. Further, it may be noted from Table 1 that at a particular value ($c = 4.25$) of food-limited parameter, all the three species will survive but in the presence of toxicant the top predator may die out. Also, it may be pointed out from Table 1 that at particular value ($Q_0 = 2.988$) of toxicant input rate all the three species will survive but if we increase the value of food-limited parameter then the top predator may die out.

From Figs.7, 8 and 9, it is observed that in all three cases of equilibria, the densities of prey and predator populations decrease in the presence of toxicant in the system. From Fig.10, it is noted that as the value of food-limited parameter increases the equilibrium levels of all the three species decrease and for a particular value ($c = 15.25$), the top predator even may die out. From Fig.11, it is observed that as the value of the death rate of intermediate predator due to toxicant increases, the equilibrium level of prey population increases due to the decrease in the intermediate predator population and the equilibrium levels of predator populations decrease and at a particular value ($\beta_{11} = 24.1$), the top predator even may die out. From Fig.12, it is observed that as the value of the death rate of top predator due to toxicant increases, the equilibrium level of prey and top predator populations decrease and intermediate predator population increases due to the decrease in top predator for which intermediate predator is prey. From Fig.13, it is noted that as the toxicant input rate into the environment increases, then the equilibrium level of all the three species decrease and for a particular value ($Q_0 = 9.988$), the predator populations

Table 1: Simulation experiments of main model for different values of parameters c, β_{11}, β_{22}, Q_0 and a_4.

Figs.	Parameter	Equilibrium Values of x y z
Fig.10	c=04.25	0.8320, 0.6076, 0.4477
	c=10.25	0.5571, 0.4525, 0.0442
	c=15.25	0.5021, 0.3647, 0.0000
Fig.11	β_{11}=01.1	0.8320, 0.6076, 0.4477
	β_{11}=16.1	1.0246, 0.5169, 0.1952
	β_{11}=24.1	1.3746, 0.3983, 0.0000
Fig.12	β_{22}=01.15	0.8320, 0.6076, 0.4477
	β_{22}=16.15	0.6934, 0.6901, 0.1455
	β_{22}=24.15	0.6751, 0.7024, 0.1042
Fig.13	Q_0=02.988	0.8320, 0.6076, 0.4477
	Q_0=06.988	0.6351, 0.4915, 0.1226
	Q_0=09.988	0.5194, 0.3398, 0.0000
	Q_0=14.988	0.3330, 0.0000, 0.0000
Fig.15	Q_0=2.988	0.8320, 0.6076, 0.4477
	Q_0=1.988	0.8851, 0.6294, 0.5417
	Q_0=0.988	0.9418, 0.6479, 0.6437
Fig.14	a_4=04.21	0.8320, 0.6076, 0.4477
	a_4=08.21	0.8181, 0.6152, 0.4258
	a_4=16.21	0.8050, 0.6224, 0.4053
Fig.16	c=04.25, β_{11}=01.1	0.8320, 0.6076, 0.4477
	c=10.25, β_{11}=16.1	0.7496, 0.3431, 0.0000
Fig.17	c=04.25, β_{22}=01.15	0.8320, 0.6076, 0.4477
	c=10.25, β_{22}=16.15	0.5374, 0.4667, 0.0000
Fig.18	c=04.25, Q_0=02.988	0.8320, 0.6076, 0.4477
	c=10.25, Q_0=06.988	0.4981, 0.3149, 0.0000
	c=15.25, Q_0=09.988	0.4305, 0.1315, 0.0000
	c=24.24, Q_0=14.988	0.3330, 0.0000, 0.0000

may even die out. On the other hand if the toxicant input rate decreases then the equilibria of all the population would increase (see Fig.15). From Fig.14, it is observed that as the toxicant transfer rate from intermediate predator to top predator increases, the equilibrium levels of prey and top predator populations decrease and intermediate predator population increases and this may happen because of the decrease in top predator population density. From Fig.16, it is noted that as the values of food-limited parameter c and the death rate of intermediate predator due to toxicant β_{11}, simultaneously increases then the equilibrium levels of all the three species decrease and for particular values of c and β_{11} ($c = 10.25$, $\beta_{11} = 16.1$), the top predator may even die out. From Fig.17, it is observed that as the values of food-limited parameter c and the death rate of top predator due to toxicant β_{22}, simultaneously increases then the equilibrium levels of all the three species decrease and for particular values of c and β_{22} ($c = 10.25$, $\beta_{22} = 16.15$), the top predator may even die out. From Fig.18, the synergistic adverse effect of food-limited parameter c and the toxicant input rate Q_0 on all the populations in the system is observed. Because, it may be noted from Table 1 that when both the parameters, *i.e.*, c and Q_0 increase then the equilibrium levels of prey, intermediate predator and top predator decrease more as compared to the equilibrium levels thus obtained after increasing c and Q_0 separately.

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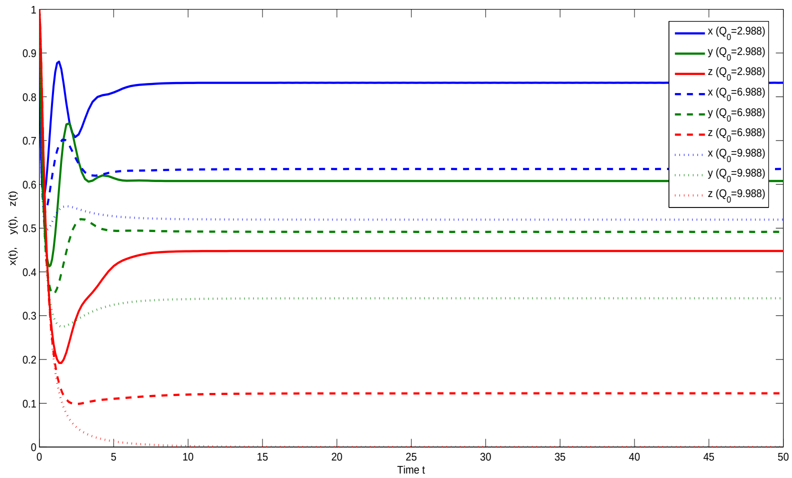

Fig.13: Variation of x, y and z with respect to time t, corresponding to increasing values of Q_0 in Main Model.

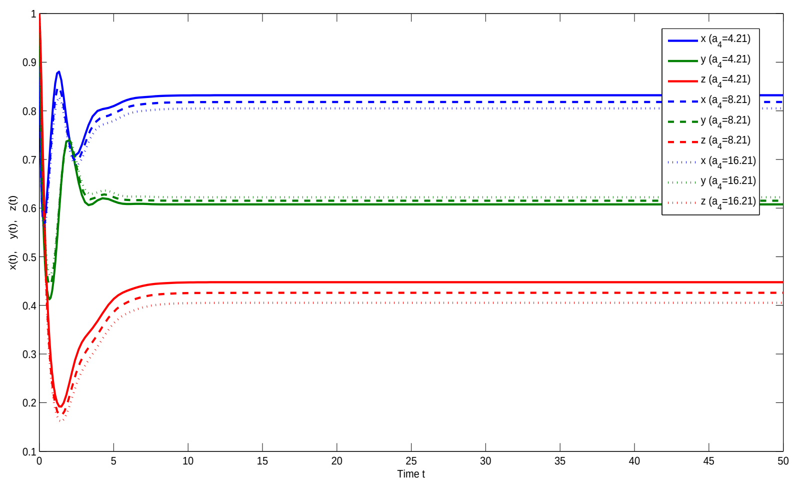

Fig.14: Variation of x, y and z with respect to time t, corresponding to different values of a_4 in Main Model.

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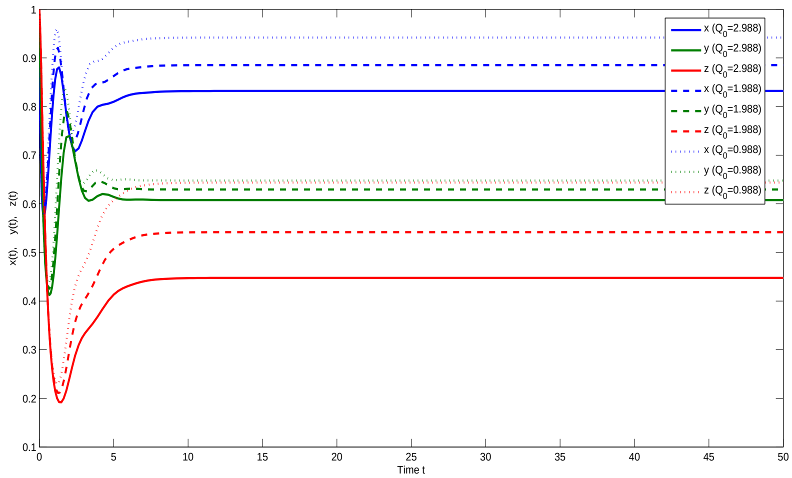

Fig.15: Variation of x, y and z with respect to time t, corresponding to decreasing values of Q_0 in Main Model.

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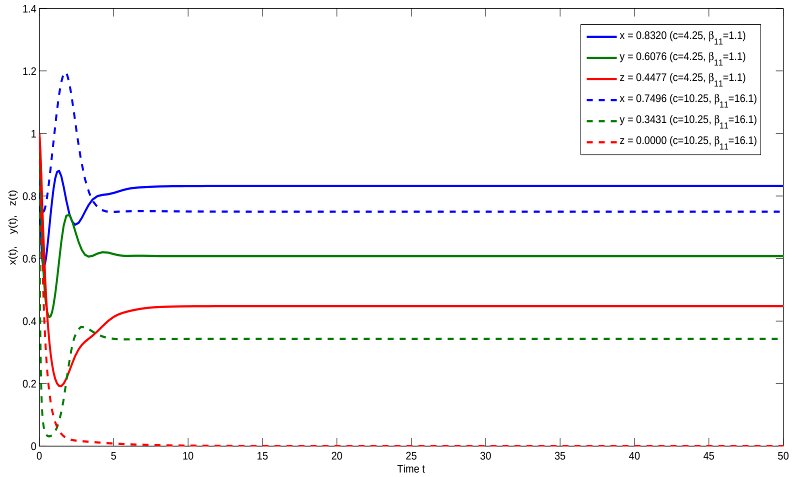

Fig.16: The values of x, y and z with respect to time t when c and β_{11} are simultaneously increased in the case of Main Model.

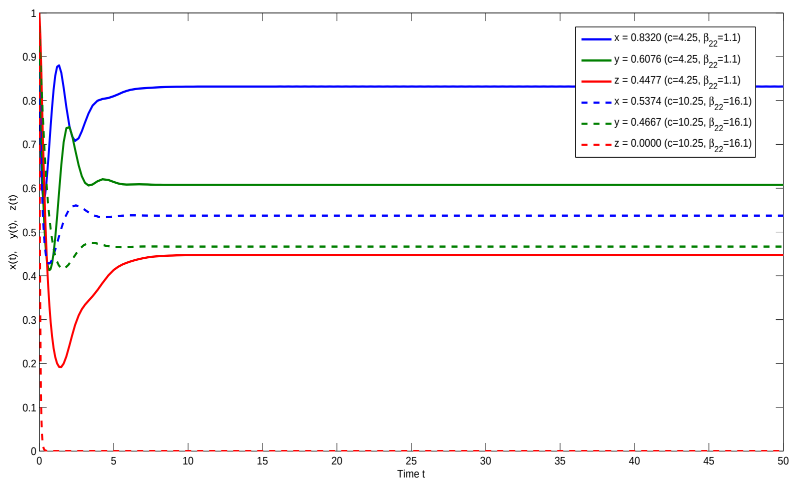

Fig.17: The values of x, y and z with respect to time t when c and β_{22} are simultaneously increased in the case of Main Model.

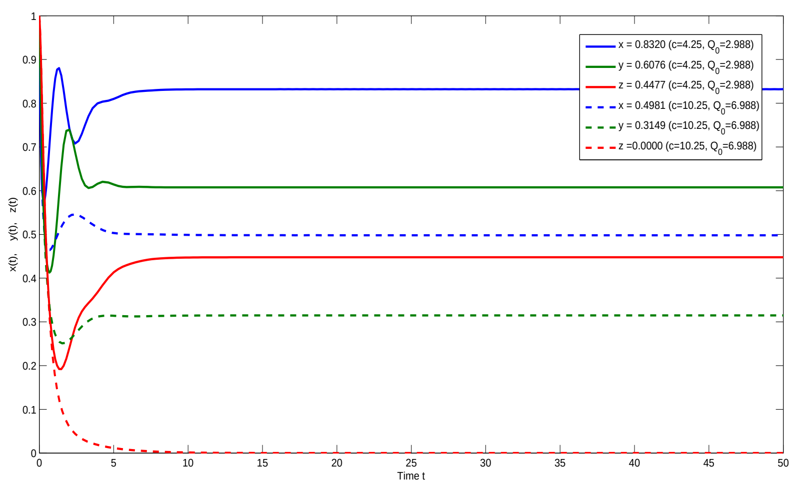

Fig.18: The values of x, y and z with respect to time t when c and Q_0 are simultaneously increased in the case of Main Model.