Global Journal of Mathematical Analysis, 2 (3) (2014) 146-151
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www.sciencepubco.com/index.php/GJMA
doi: 10.14419/gjma.v2i3.2989
Research paper

Majorization problems for $p-$valently meromorphic functions of complex order involving certain integral operator

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Abstract

The main object of this paper is to investigate the problem of majorization of certain class of meromorphic p-valent functions of complex order involving certain integral operator. Moreover we point out some new or known consequences of our main result.

Keywords: *Meromorphic functions, Starlike functions, Convex functions, Majorization problems, Hadamard product (convolution), Integral operator.*

2010 AMS Subject Classification:Primary 30C45; Secondary 30C50

1 Introduction

Let f and g are analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Due to MacGregor [8], (also see [7]) we say that f is majorized by g in Δ and we write

$$f(z) \ll g(z), (z \in \Delta) \tag{1}$$

if there exists a function ϕ, analytic in Δ, such that

$$|\phi(z)| < 1 \text{ and } f(z) = \phi(z)g(z), \quad z \in \Delta. \tag{2}$$

It may be noted here that (1) is closely related to the concept of quasi-subordination between analytic functions.
Also we say that f is subordinate to g denoted by $f \prec g$ (see [9]), if there exists a Schwarz function ω which is analytic in Δ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \Delta$, such that

$$f(z) = g(\omega(z)), z \in \Delta.$$

We denote this subordination by $f \prec g$. Furthermore, if the function g is univalent in Δ, we have

$$f \prec g \Longleftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

Denote by $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ the class of starlike and convex functions of complex order $\gamma(\gamma \in \mathbb{C} \setminus \{0\})$, satisfying the following conditions

$$\frac{f(z)}{z} \neq 0 \text{ and } \Re\left(1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right) > 0$$

and

$$f'(z) \neq 0 \text{ and } \Re\left(1 + \frac{1}{\gamma}\left[\frac{zf''(z)}{f'(z)}\right]\right) > 0, (z \in \Delta)$$

respectively. Further,

$$\mathcal{S}^*((1-\delta)cos\lambda\ e^{-i\lambda}) = S^*(\delta, \lambda),\ |\lambda| < \frac{\pi}{2};\ 0 \leq \delta \leq 1$$

the class of $\lambda-$ spiral-like function of order δ investigated by Libera [4] and

$$\mathcal{S}^*(cos\lambda\ e^{-i\lambda}) = S^*(\lambda),\ |\lambda| < \frac{\pi}{2};$$

the class of spiral-like functions introduced by Spacek [10](also see [11]).

A mojorization problem for the class of analytic starlike functions have been investigated by MacGregor [8] and Altintas et al. [1]. Recently Goyal and Goswami [3] extended these results for the class of meromorphic functions making use of certain integral operator.

Let Σ_p be the class of $p-$valently meromorphic functions which are analytic and univalent in the punctured unit disk

$$\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \Delta \setminus \{0\}$$

of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p}. \tag{3}$$

with a simple pole at the origin.

Due to Aqlan et al. [2] (see [5]), we recall the integral operator $\mathcal{J}^{\alpha}_{\beta,p}$ for meromorphic functions $f \in \Sigma_p$ as given below,

$$\mathcal{J}^{\alpha}_{\beta,p} : \Sigma_p \to \Sigma_p$$

$$\mathcal{J}^{\alpha}_{\beta,p} f(z) = \begin{pmatrix} \alpha+\beta-1 \\ \beta-1 \end{pmatrix} \frac{1}{z^{p+\beta}} \int_0^z (1-\frac{t}{z})^{\alpha-1}\ t^{\beta+p-1} f(t)\ dt \tag{4}$$

$$\mathcal{J}^{\alpha}_{\beta,p} f(z) = \begin{cases} f(z) & \alpha = 0, \beta > -1, p \in \mathbb{N}, f \in \Sigma_p \\ \frac{1}{z^p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum\limits_{n=1}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(n+\alpha+\beta)} a_{n-p} z^{n-p}, & \alpha > 0, \beta > -1, p \in \mathbb{N}, f \in \Sigma_p. \end{cases} \tag{5}$$

The following relation for $\mathcal{J}^{\alpha}_{\beta,p} f(z)$ can be obtained by simple calculation,

$$z(\mathcal{J}^{\alpha}_{\beta,p} f(z))' = (\alpha+\beta-1)\mathcal{J}^{\alpha-1}_{\beta,p} f(z) - (\alpha+\beta+p-1)\mathcal{J}^{\alpha}_{\beta,p} f(z). \tag{6}$$

Using (6), the below recurrence relation for $\mathcal{J}^{\alpha}_{\beta,p} f(z)$ can be obtained easily,

$$z(\mathcal{J}^{\alpha}_{\beta,p} f(z))^{(q+1)} = (\alpha+\beta-1)(\mathcal{J}^{\alpha-1}_{\beta,p} f(z))^{(q)} - (\alpha+\beta+p+q-1)(\mathcal{J}^{\alpha}_{\beta,p} f(z))^{(q)}. \tag{7}$$

In the present paper we investigate a majorization problem for the class of $p-$valently meromorphic starlike functions of complex order associated with the generalized integral operator due to Aqlan [2] and Murugusundaramoorthy and Magesh [6].

Definition 1.1. *A function $f(z) \in \Sigma_p$ is said to in the class $\mathcal{M}^{p,q}_{\alpha,\beta}(\gamma, A, B)$ of meromorphic functions of complex order $\gamma \neq 0$ in Δ^* if and only if*

$$1 - \frac{1}{\gamma}\left[\frac{z(\mathcal{J}^{\alpha}_{\beta,p} f(z))^{(q+1)}}{(\mathcal{J}^{\alpha}_{\beta,p} f(z))^{(q)}} + p + q\right] \prec \frac{1+Az}{1+Bz}, \tag{8}$$

where $z \in \Delta^*, p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta > -1, \alpha > 0, \gamma \in \mathbb{C} \setminus \{0\}$ and $-1 \leq B < A \leq 1$.

For simplicity, we put

$$\mathcal{M}^{p,q}_{\alpha,\beta}(\gamma,1,-1)=\mathcal{M}^{p,q}_{\alpha,\beta}(\gamma),$$

where $\mathcal{M}^{p,q}_{\alpha,\beta}(\gamma)$ denote the class of functions $f\in\Sigma_p$ satisfying the following inequality:

$$\Re\left(1-\frac{1}{\gamma}\left[\frac{z(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q+1)}}{(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q)}}+p+q\right]\right)>0 \tag{9}$$

where $z\in\Delta^*, p,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, \beta>-1, \alpha>0, \gamma\in\mathbb{C}\setminus\{0\}$.

Example 1.2. *Putting* $\gamma=(p-\delta)cos\lambda\ e^{-i\lambda},\ |\lambda|<\frac{\pi}{2};\ 0\le\delta<p$ *the class*

$$\mathcal{M}^{p,q}_{\alpha,\beta}(\gamma)=\mathcal{M}^{p,q}_{\alpha,\beta}((p-\delta)cos\lambda\ e^{-i\lambda})\equiv\mathcal{M}^{p,q}_{\alpha,\beta}(\delta,\lambda)$$

called the generalized class of $\lambda-$*spiral-like functions of order* $\delta(0\le\delta<p)$ *if*

$$\Re\left(e^{i\lambda}\left[\frac{z(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q+1)}}{(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q)}}+q\right]\right)<-\delta\ cos\lambda \tag{10}$$

where $z\in\Delta^*, p,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, \beta>-1, \alpha>0, \gamma\in\mathbb{C}\setminus\{0\}$.

Example 1.3. *Putting* $\gamma=(p-\delta); 0\le\delta<p$ *the class* $\mathcal{M}^{p,q}_{\alpha,\beta}(p-\delta)\equiv\mathcal{M}^{p,q}_{\alpha,\beta}(\delta)$, *the generalized class of* $p-$*valently meromorphic starlike functions of order* $\delta(0\le\delta<p)$ *if*

$$\Re\left(\frac{z(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q+1)}}{(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q)}}+q\right)<-\delta \tag{11}$$

where $z\in\Delta^*, p,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, \beta>-1, \alpha>0, \gamma\in\mathbb{C}\setminus\{0\}$.

Remark 1.4. *By taking* $q=0$ *in Example 1.3,* $\mathcal{M}^{p,0}_{\alpha,\beta}(p-\delta)\equiv\mathcal{M}^{p}_{\alpha,\beta}(\delta)$ *the class of* $p-$*valently meromorphic starlike functions of order* $\delta(0\le\delta<p)$ *if*

$$\Re\left(\frac{z(\mathcal{J}^{\alpha}_{\beta,p}f(z))'}{(\mathcal{J}^{\alpha}_{\beta,p}f(z))}\right)<-\delta$$

where $z\in\Delta^*, \beta>-1, \alpha>0, \gamma\in\mathbb{C}\setminus\{0\}$.

2 Majorization problem for the class $\mathcal{M}^{p,q}_{\alpha,\beta}(\gamma,A,B)$

Theorem 2.1. *Let the function* $f\in\Sigma_p$ *and* $g\in\mathcal{M}^{p,q}_{\alpha,\beta}(\gamma,A,B)$ *if* $(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q)}$ *is majorized by* $(\mathcal{J}^{\alpha}_{\beta,p}g(z))^{(q)}$ *in* Δ^* *then*

$$|(\mathcal{J}^{\alpha-1}_{\beta,p}f(z))^{(q)}|\le|(\mathcal{J}^{\alpha-1}_{\beta,p}g(z))^{(q)}|,\ \ |z|\le r_1, \tag{12}$$

$r_1=r_1(A,B,\alpha,\beta,\gamma,\rho)$ *is the smallest positive root of the equation*

$$\begin{aligned}&|(\alpha+\beta-1)B-\gamma(A-B)|\ r^3-\{(\alpha+\beta-1)+2\rho|B|\ \}\ r^2\\ -\ &\{|(\alpha+\beta-1)B-\gamma(A-B)|+2\rho\}\ r+(\alpha+\beta-1)=0,\end{aligned} \tag{13}$$

where $z\in\Delta^*, p,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, \beta>-1, \alpha>0, \gamma\in\mathbb{C}\setminus\{0\}$ *and* $-1\le B<A\le1$.

Proof. Since $g(z)\in\mathcal{M}^{p,q}_{\alpha,\beta}(\gamma,A,B)$, we readily obtain from (8) that, if

$$1-\frac{1}{\gamma}\left[\frac{z(\mathcal{J}^{\alpha}_{\beta,p}g(z))^{(q+1)}}{(\mathcal{J}^{\alpha}_{\beta,p}g(z))^{(q)}}+p+q\right]=\frac{1+Aw(z)}{1+Bw(z)} \tag{14}$$

where w denotes the well known class of bounded analytic functions in Δ and

$$w(0)=0\quad\text{and}\quad|w(z)|\le|z|,\quad(z\in\Delta). \tag{15}$$

From (14),we get

$$\frac{z(\mathcal{J}_{\beta,p}^{\alpha}g(z))^{(q+1)}}{(\mathcal{J}_{\beta,p}^{\alpha}g(z))^{(q)}} = -\frac{(p+q)+[(p+q)B+\gamma(A-B)]w(z)}{1+Bw(z)}. \tag{16}$$

Using (7) in the above equation, we get,

$$(\mathcal{J}_{\beta,p}^{\alpha}g(z))^{(q)} = \frac{(\alpha+\beta-1)[1+Bw(z)]}{(\alpha+\beta-1)+[(\alpha+\beta-1)B-\gamma(A-B)]\ w(z)}(\mathcal{J}_{\beta,p}^{\alpha-1}g(z))^{(q)}. \tag{17}$$

Hence, by making use of (15), we get,

$$|(\mathcal{J}_{\beta,p}^{\alpha}g(z))^{(q)}| \leq \frac{(\alpha+\beta-1)[1+|B|\ |z|]}{(\alpha+\beta-1)-|(\alpha+\beta-1)B-\gamma(A-B)|\ |z|}|(\mathcal{J}_{\beta,p}^{\alpha-1}g(z))^{(q)}|. \tag{18}$$

Since $(\mathcal{J}_{\beta,p}^{\alpha}f(z))^{(q)}$ is majorized by $(\mathcal{J}_{\beta,p}^{\alpha}g(z))^{(q)}$ in Δ^* from (2), we have

$$(\mathcal{J}_{\beta,p}^{\alpha}f(z))^{(q)} = \phi(z)(\mathcal{J}_{\beta,p}^{\alpha}g(z))^{(q)}$$

Differentiating the above equation w.r.t z and multiplying by z, we have,

$$z(\mathcal{J}_{\beta,p}^{\alpha}f(z))^{(q+1)} = z\phi'(z)(\mathcal{J}_{\beta,p}^{\alpha}g(z))^{(q)} + z\phi(z)(\mathcal{J}_{\beta,p}^{\alpha}g(z))^{(q+1)}.$$

By using (7), we get,

$$(\mathcal{J}_{\beta,p}^{\alpha-1}f(z))^{(q)} = \frac{z}{\alpha+\beta-1}\phi'(z)(\mathcal{J}_{\beta,p}^{\alpha}g(z))^{(q)} + \phi(z)(\mathcal{J}_{\beta,p}^{\alpha-1}g(z))^{(q)}. \tag{19}$$

Noting that the Schwarz function $\phi(z)$ satisfies

$$|\phi^{'}(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} \tag{20}$$

and using (18) and (20) in (19) we have

$$|(\mathcal{J}_{\beta,p}^{\alpha-1}f(z))^{(q)}| \leq \left(|\phi(z)| + \tfrac{(1-|\phi(z)|^2)}{(1-|z|^2)}\cdot\tfrac{|z|\ (1+|B|\ |z|)}{(\alpha+\beta-1)-|(\alpha+\beta-1)B-\gamma(A-B)|\ |z|}\right)|(\mathcal{J}_{\beta,p}^{\alpha-1}g(z))^{(q)}|$$

which upon setting

$$|z| = r \text{ and } |\phi(z)| = \rho, \quad (0 \leq \rho \leq 1)$$

leads us to the inequality

$$|(\mathcal{J}_{\beta,p}^{\alpha-1}f(z))^{(q)}| \leq \frac{\theta(\rho)}{(1-r^2)\{(\alpha+\beta-1)-|(\alpha+\beta-1)B-\gamma(A-B)|r\}}|(\mathcal{J}_{\beta,p}^{\alpha-1}g(z))^{(q)}|, \tag{21}$$

where

$$\theta(\rho) = \rho(1-r^2)\{\ (\alpha+\beta-1)-|(\alpha+\beta-1)B-\gamma(A-B)|\ r\} + (1-\rho^2)(1+|B|\ r)r$$

takes its maximum value at $\rho = 1$. Furthermore, if $0 \leq \sigma \leq r_1$, the function $\varphi(\rho)$ defined by

$$\varphi(\rho) = \rho(1-\sigma^2)\{\ (\alpha+\beta-1)-|(\alpha+\beta-1)B-\gamma(A-B)|\ \sigma\} + (1-\rho^2)(1+|B|\ \sigma)\sigma$$

is an increasing function on $(0 \leq \rho \leq 1)$ so that

$$\varphi(\rho) \leq \varphi(1) = (1-\sigma^2)\{\ (\alpha+\beta-1)-|(\alpha+\beta-1)B-\gamma(A-B)|\ \sigma\}. \tag{22}$$

Therefore, from this fact, (21) gives the inequality (12). □

3 Corollaries and Concluding Remarks

By taking $A = 1; B = -1$ and $\rho = 1$ in Theorem 2.1, we state the following corollary without proof.

Corollary 3.1. *Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}^{p,q}_{\alpha,\beta}(\gamma)$ if $(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q)}$ is majorized by $(\mathcal{J}^{\alpha}_{\beta,p}g(z))^{(q)}$ in Δ^* then*

$$|(\mathcal{J}^{\alpha-1}_{\beta,p}f(z))^{(q)}| \leq |(\mathcal{J}^{\alpha-1}_{\beta,p}g(z))^{(q)}|, \ \ |z| \leq r_2,$$

where $r_2 = r_2(\alpha, \beta, \gamma)$ is the smallest positive root of the equation

$$\{|(\alpha+\beta-1)+2\gamma|r^3 - \{\alpha+\beta+1\}r^2 - \{|(\alpha+\beta-1)+2\gamma|+2\}r + (\alpha+\beta-1) = 0, \quad given \ \ by$$

$$r_2 = \frac{L_1 - \sqrt{L_1^2 - 4|\alpha+\beta-1+2\gamma|(\alpha+\beta-1)}}{2|\alpha+\beta-1+2\gamma|}$$

and $L_1 = \alpha + \beta + 1 + |\alpha+\beta-1+2\gamma|$.

By setting $\alpha = 1$ in Corollary 3.1, we state the following corollary.

Corollary 3.2. *Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}^{p,q}_{\alpha,\beta}(\gamma)$ if $(\mathcal{J}^{1}_{\beta,p}f(z))^{(q)}$ is majorized by $(\mathcal{J}^{1}_{\beta,p}g(z))^{(q)}$ in Δ^* then*

$$|(f(z))^{(q)}| \leq |(g(z))^{(q)}|, \ \ |z| \leq r_3,$$

where $r_3 = r_3(1, \beta, \gamma)$ is the smallest positive root of the equation

$$|\beta+2\gamma|r^3 - (\beta+2)r^2 - \{|\beta+2\gamma|+2\}r + \beta = 0, \quad given \ \ by$$

$$r_3 = \frac{L_2 - \sqrt{L_2^2 - 4\beta|\beta+2\gamma|}}{2|\beta+2\gamma|}$$

and $L_2 = \beta + 2 + |\beta+2\gamma|$.

By setting $\alpha = 1, \beta = 1$ and $\gamma = p - \delta$ in Corollary 3.1, we state the following corollary.

Corollary 3.3. *Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}^{p,q}_{\alpha,\beta}(\delta)$ if $(\mathcal{J}^{1}_{1,p}f(z))^{(q)}$ is majorized by $(\mathcal{J}^{1}_{1,p}g(z))^{(q)}$ in Δ^* then*

$$|(f(z))^{(q)}| \leq |(g(z))^{(q)}|, \ \ |z| \leq r_4,$$

where $r_4 = r_4(1, 1, (p-\delta)1)$ is the smallest positive root of the equation

$$|1+2(p-\delta)|r^3 - 3r^2 - \{|1+2(p-\delta)|+2\}r + 1 = 0, \quad given \ \ by$$

$$r_4 = \frac{L_3 - \sqrt{L_3^2 - 4|1+2(p-\delta)|}}{2|1+2(p-\delta)|}$$

and $L_3 = 3 + |1+2(p-\delta)|$.

Remark 3.4. *By taking $p = 1$ and $q = 0$, Corollary 3.3 yields results of Goyal and Gosami[3].*

By taking $\gamma = (p-\delta)cos\ \lambda e^{-i\lambda}$ $(|\lambda| < \frac{\pi}{2}, \delta(0 \leq \delta < p)$, in Corollary 3.1, we state the following corollary without proof.

Corollary 3.5. *Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}^{p,q}_{\alpha,\beta}(\alpha, \lambda)$ if $(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q)}$ is majorized by $(\mathcal{J}^{\alpha}_{\beta,p}g(z))^{(q)}$ in Δ^* then*

$$|(\mathcal{J}^{\alpha}_{\beta,p}f(z))^{(q)}| \leq |\mathcal{J}^{\alpha}_{\beta,p}(g(z))^{(q)}|, \ |z| \leq r,$$

where $r = r(T, \lambda)$ is given by

$$r = \frac{T - \sqrt{T^2 - 4|\alpha+\beta-1+2(p-\delta)cos\lambda\ e^{-i\lambda}|(\alpha+\beta-1)}}{2|\alpha+\beta-1+2(p-\delta)cos\lambda\ e^{-i\lambda}|}$$

and

$$T = (\alpha+\beta+1) + |1+2(p-\delta)cos\lambda e^{-i\lambda}|.$$

Concluding Remarks: Further specializing the parameters α, β one can define the various other interesting subclasses of Σ_p involving the various integral operators and the corresponding corollaries as mentioned above can be derived easily.

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