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Research Paper

Generalized Fibonacci – Lucas sequence its Properties

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Abstract

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci sequences are a source of many nice and interesting identities. A similar interpretation exists for Lucas sequence. The Fibonacci number, Lucas numbers and their generalization have many interesting properties and applications to almost every field. Fibonacci sequence is defined by the recurrence formula $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ and $F_0 = 0, F_1 = 1$, where F_n are an n^{th} number of sequences. The Lucas Sequence is defined by the recurrence formula $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$ and $L_0=2, L_1=1$, where L_n an nth number of sequences are. In this paper, we present generalized Fibonacci-Lucas sequence that is defined by the recurrence relation $B_n = B_{n-1} + B_{n-2}$, $n \geq 2$ with $B_0 = 2s$, $B_1 = s$. We present some standard identities and determinant identities of generalized Fibonacci-Lucas sequences by Binet's formula and other simple methods.

Keywords: *Fibonacci sequence, Lucas Sequence, Generalized Fibonacci sequence, Binet's Formula.*

1. Introduction

The Fibonacci and Lucas sequences are well-known examples of second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. As illustrate in the tome by Koshy [5] the Fibonacci and Lucas number are arguable two of the most interesting sequence in all of mathematics. Many identities have been documented in an extensive list that appears in the work of Vajda [12], where they are proved by algebra means, even though combinatorial proof of many of these interesting identities. We introduced Generalized Fibonacci-Lucas Sequence and its Properties Fibonacci numbers, Lucas number's and their generalization have many interesting Properties and application to almost every field. The Fibonacci sequence [5] is a sequence of numbers starting with integer 0 and 1, where each next term of the sequence calculated as the sum of the previous two.

i.e., $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$,and $F_0 = 0, F_1 = 1$ (1.1)

The similar interpretation also exists for Lucas sequence. Lucas sequence [7] is defined by the recurrence relation,

$L_n = L_{n-1} + L_{n-2}$, $n \geq 2$,and $L_0=2, L_1=1$. (1.2)

In this paper, we present various properties of the Generalized Fibonacci-Lucas

Sequence $\{B_n\}$ defined by $B_n = B_{n-1} + B_{n-2}$ $n \geq 2$ and $B_0 = 2b, B_1 = s$ (1.3)

The Binet's formula for Fibonacci sequence is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n\right\} \tag{1.4}$$

Where $\alpha = \dfrac{1+\sqrt{5}}{2} =$ Golden ratio = 1.618

And $\beta = \dfrac{1-\sqrt{5}}{2} =$ Golden ratio = -0.618

Similarly, the Binet's formula for Lucas sequence is given by

$$L_n = \alpha^n + \beta^n = \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1+\sqrt{5}}{2} \right)^n \right\}$$

2. Preliminary results generalized Fibonacci - Lucas sequence

We need to introduce some basic results of Generalized Fibonacci-Lucas sequence
Generalized Fibonacci-Lucas sequence $\{B_n\}$ is defined by recurrence relation.

$B_n = B_{n-1} + B_{n-2}$, $n \geq 2$ (2.1)

With initial conditions $B_0 = 2s$ and $B_1 = s$ the associated initial Condition B_0 and B_1 are the sum of initial condition of generalized Fibonacci-Lucas sequence respectively.

i.e. $F_0 + b\, L_0 = B_0$ and $s\,(F_1+L_1) = 2B_1$ (2.2)

The few terms of above sequence are 2b, s, 2b+s, 2b+2s, 4b+3s, 6b+5s and so on. The relation between Fibonacci sequence and Generalized Fibonacci-Lucas Sequence can be written as $B_n = F_n + bL_n, n \geq 0$.

The recurrence relation (1) has the characteristic equation $x^2 - x + 1 = 0$ this has two roots $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$

Now notice a few things about α and β

$\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$ and $\alpha\beta = -1$

Using these two roots, we obtain Binet's recurrence relation

$$B_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} + b\left(\alpha^n + \beta^n\right)$$

$$= \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1+\sqrt{5}}{2} \right)^n \right\} + b \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}$$

3. Generating function

Now we state derive generating function of generalized Fibonacci-Lucas sequence

Generating from is $\sum_{n=0}^{\infty} B_n x^n = \dfrac{2b + (s-2b)x}{(1 - x - x^2)}$ (3.1)

Let's apply power series to sequence $\{B_n\}$

Let 2b + sx + (2b+s) x^2 +................= $\sum_{n=0}^{\infty} B_n x^n$ Where B_n is n^{th} term of sequence $\{B_n\}$

This is called generating series of Generalized Fibonacci - Lucas Sequence $\{B_n\}$

Now multiplying the generating series by the Polynomial

$$(1 - x - x^2) \sum_{n=0}^{\infty} B_n x^n = \sum_{n=0}^{\infty} B_n x^n - \sum_{n=0}^{\infty} B_n x^{n+1} - \sum_{n=0}^{\infty} B_n x^{n+2}$$

$$= (B_0 + B_1 x + \sum_{n=2}^{\infty} B_n x^n) - (B_0 x + \sum_{n=2}^{\infty} B_{n-1} x^n) - \sum_{n=2}^{\infty} B_{n-2} x^n$$

$$=B_0 + (B_1-B_0)\ x + \sum_{n=2}^{\infty} (B_n - B_{n-1} - B_{n-2})x^n$$

$$= 2b + (s-2b)\ x + \sum_{n=2}^{\infty} (B_{n-1} + B_{n-2} - B_{n-1} - B_{n-2})x^n$$

$$= 2b + (s-2b)\ x + \sum_{n=2}^{\infty} (0)x^n$$

$$= 2b + (s-2b)\ x$$

Therefore, $(1- x - x^2) \sum_{n=0}^{\infty} B_n x^n = 2b + (s-2b)x$

Hence $\sum_{n=0}^{\infty} B_n x^n = \dfrac{2b + (s-2b)x}{(1-x-x^2)}$

4. Properties of generalized Fibonacci- Lucas sequence

Despite its simple appearance the Generalized Fibonacci-Lucas sequence $\{B_n\}$ contains a wealth of subtle and fascinating properties [4], [6], [9], [12].

Sum of First n terms:
Theorem 4.1: *Let B_n is the n^{th} Fibonacci-Like number, and then Sum of the first n terms of generalized Fibonacci - Lucas sequence is*

$$(B_1+B_2+B_3+........B_n) = \sum_{k=1}^{n} B_k = B_{n+2} - s \tag{4.1}$$

Proof: we know that the follows relation holds:
$B_1 = B_3 - B_2$
$B_2 = B_4 - B_3$ (Since $B_3 = B_2 + B_1$)
$B_3 = B_5 - B_4$
$B_{n-1} = B_{n+1} - B_n$
$B_n = B_{n+2} - B_{n+1}$
Term wise addition of all above equations, we obtain
$(B_1+B_2+B_3+..........B_n)= B_{n+2} - B_2$
$= B_{n+2} - s$

Sum of First n terms with even indices:
Theorem 4.2: *Let B_n be the n^{th} Fibonacci-Lucas sequence, then Sum of the first n terms with even indices is*

$$(B_2+B_4+B_6+..........B_{2n}) = \sum_{k=1}^{n} B_{2k} = B_{2n+1} - s \tag{4.2}$$

Sum of First n terms with square indices:
Theorem 4.3: *Let B_n be the n^{th} Fibonacci-Lucas sequence, then Sum of the square of first n terms is*

$$(B_1^2 + B_2^2 + B_3^2 + ...B_n^2) = \sum_{k=1}^{n} B_k^2 = B_n B_{n-2} \tag{4.3}$$

Sum of First n terms with odd indices:
Theorem 4.4: *Let B_n be the n^{th} Fibonacci-Lucas sequence, then Sum the first n terms with odd indices is*

$$\left(B_1 + B_3 + B_5 + B_7 + ...+ B_{2n-1}\right) = \sum_{k=1}^{n} B_{2k-1} = B_{2n} - B_{2n-2} \tag{4.4}$$

Now we state and prove some nice identities similar to those obtained for Fibonacci and Lucas sequences [1], [2], [4], and [12]

5. Some Identities generalized Fibonacci- Lucas sequences

In this section, some identities of Generalized Fibonacci-Lucas sequence are presented which can be easily derived by explicit sum formula and generating function.

Explicit Sum Formula:

Theorem 5.1: *The explicit sum formula for Generalized Fibonacci-Lucas sequence is given by for positive integer n, prove that*

$$B_{2n} = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} \tag{5.1}$$

Proof: By equation (2.1), it follows that

$B_{2n} = B_{2n-1} + B_{2n-2}$

$= (B_{2n-2} + B_{2n-3}) + (B_{2n-3} + B_{2n-4})$

$= B_{2n-2} + 2B_{2n-3} + B_{2n-4}$

$= (B_{2n-3} + B_{2n-4}) + 2(B_{2n-4} + B_{2n-5}) + (B_{2n-5} + B_{2n-6})$

$= B_{2n-3} + 3B_{2n-4} + 3B_{2n-5} + B_{2n-6}$

$= B_0 + nB_1 + \frac{n(n-1)}{2} B_2 + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{n(n-1)}{2} B_{n-2} + nB_{n-1} + B_n$

$$\Rightarrow B_{2n} = \sum_{m=0}^{n} \binom{n}{m} B_{n-m}$$

Hence $B_{2n} = \sum_{m=0}^{n} \binom{n}{m} B_{n-m}$

Theorem 5.2: *The explicit sum formula for Generalized Fibonacci-Lucas sequence is given by for positive integer n,*

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_{n-2k} \tag{5.2}$$

Theorem 5.3: For every positive integer n, prove that

$$B_{m+1}B_n - B_{m+1}B_{n+1} = (-1)^n B_{m+1}B_{n-m-1}, n \geq 1 \tag{5.3}$$

Proof: Let n be fixed and we Proved by inducting on m.

When m = 0, then

$B_1B_n - B_1 B_{n+1} = (-1)^1 B_1 B_{n-1}$

$s B_n - s B_{n+1} = - s B_{n-1}$

$s (B_n - B_{n+1}) = - s B_{n-1}$

$s (-B_{n-1}) = - s B_{n-1}$

$- s B_{n-1} = s (1+s) B_{n-1}$

Which is true?

When m=1, then

$B_{1+1} B_n - B_{1+1} B_{n+1} = (-1)^1 B_{1+1} B_{n-1-1}$

$B_2 B_n - B_2 B_{n+1} = (-1)^1 B_2 B_{n-2}$

$B_2 (B_n-B_{n+1}) = (-1)^1 B_2 B_{n-2}$

$(2b+s) (B_n-B_{n+1}) = (-1)^1 (2b+s) B_{n-2}$

$(2b+s) (-B_{n-2}) = - (2b+s) B_{n-2}$

$- (2b+s) B_{n-2} = - (2b+s) B_{n-2}$

Which also is true?

Now assume that identity is true for m = k+1, then by assumption

$B_k B_n - B_k B_{n+1} = (-1)^n B_k B_{n-k}$ (5.4)

$B_{k-1} B_n - B_{k-1} B_{n+1} = (-1)^n B_{k-1} B_{n-k+1}$ (5.5)

Adding equation (5.4) and (5.5), we get

$B_k B_n + B_{k-1} B_n - B_k B_{k+1} - B_{k-1} B_{n+1} = (-1)^n B_k B_{n-k} + (-1)^n B_{n-k+1}$

$(B_k + B_{k-1}) B_n - (B_k + B_{k-1}) B_{n+1} = (-1)^n (B_k B_{n-k} + B_{n-k+1})$

$B_{k+1} B_n - B_{k+1} B_{n+1} = (-1)^n B_{k+1} B_{n-k-1}$

Which is precisely our identity when k = m

Hence $B_{m+1}B_n - B_{m+1}B_{n+1} = (-1)^n B_{m+1}B_{n-m-1}, n \geq 1$

Theorem 5.4: *For every positive integer n, prove that*

$B_{2n} = B_{2n+1} - B_{2n-1}$ (5.6)

Proof: we shall have proved this identity by induction matched over n.
For n=0
$B_{2xo} = B_{2x0+1} - B_{2x0-1}$
$B_o = B_1 - B_{-1}$
2b=s - (s – 2b)
2b= s – s +2b
2b = 2b
Which is also true for n=0
When n = 1 than
$B_{2x1} = B_{2x1+1} - B_{2x1-1}$
$B_2 = B_3 - B_1$
(2b+s) = 2b+2s – s
2b+s = 2b+ s which is also true for n = 1
For n = k
$B_{2k} = B_{2k+1} - B_{2k-1}$
For n = k which is also true.
Now assume that identity is true for n =1, 2, 3....k and
We so that it holds:
For n = k+1, then by assumption
$B_{2(k+1)} = B_{2(k+1)+1} - B_{2(k+1)-1}$
$B_{2k+2} = B_{2k+3} - B_{2k+1}$
$= (B_{2k+2} + B_{2k+1}) - B_{2k+1}$
$= B_{2k+2} + B_{2k+1} - B_{2k+1}$
$= B_{2k+2}$
Which is also true, for n = k+1
Hence, the result is true for all.

Theorem 5.5: For every positive integer n, prove that

$sF_{n-1}= B_n-B_{n-2} \qquad n \geq 2$ (5.7)

Proof: we shall Prove this identity by induction over n, for n= 2
$s F_{n-2} = sF_{2-1} = s F_1$
= s .1
= s
$= B_2-B_0$
Now suppose that identity hold for n =k-1, n = k-2
Then,
$sF_{k-2} = B_{k-1} - B_{k-3}$ (5.8)
$s F_{k-3} = B_{k-2} - B_{k-4}$ (5.9)
On adding equation (5.8) & (5.9) we get,
$s F_{k-2} + s B_{k-3}= (B_{k-1} + B_{k-2}) - (B_{k-3} + B_{k-4})$
$s (F_{k-2} + B_{k-3}) = B_k - B_{k-2}$
$s F_{k-1}= B_k - B_{k-2}$
Which is true for n = k,
$s F_{n-1}= B_n-B_{n-2} \qquad n \geq 2$

Theorem 5.6: For every positive integer n,

$$B_3 + B_6 + B_9 + ...+ B_{3n} = \frac{1}{2}\left[B_{3n+2} - (2b+s)\right] \tag{5.10}$$

Proof: By using Binet's formula, we have

$$B_3 + B_6 + B_9 + ...+ B_{3n}$$
$$= \frac{\alpha^3 - \beta^3}{\sqrt{5}} + b(\alpha^3 - \beta^3) + \frac{\alpha^6 - \beta^6}{\sqrt{5}} + b(\alpha^6 - \beta^6) + ... + \frac{\alpha^{3n} - \beta^{3n}}{\sqrt{5}} + b(\alpha^{3n} - \beta^{3n})$$
$$= \frac{1}{\sqrt{5}}\left[(\alpha^3 + \alpha^6 + \alpha^9 .....\alpha^{3n}) - (\beta^3 + \beta^6 + \beta^9 .....\beta^{3n})\right] + b\left[(\alpha^3 + \alpha^6 + \alpha^9 .....\alpha^{3n}) - (\beta^3 + \beta^6 + \beta^9 .....\beta^{3n})\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{\alpha^{3n+3}-\alpha^{3}}{\alpha^{3}-1}\right)-\left(\frac{\beta^{3n+3}-\beta^{3}}{\beta^{3}-1}\right)\right]+b\left[\frac{\alpha^{3n+3}-\alpha^{3}}{\alpha^{3}-1}+\frac{\beta^{3n+3}-\beta^{3}}{\beta^{3}-1}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\frac{\alpha^{3n+2}-\alpha^{2}}{2}-\left(\frac{\beta^{3n+2}-\beta^{2}}{2}\right)\right]+b\left[\frac{\alpha^{3n+2}-\alpha^{2}}{2}+\frac{\beta^{3n+2}-\beta^{2}}{2}\right]$$

$$=\frac{1}{2}\left[\frac{\alpha^{3n+2}-\beta^{3n+2}}{\sqrt{5}}+b\left(\alpha^{3n+2}+\beta^{3n+2}\right)\right]+b\left[\frac{\alpha^{2}+\beta^{2}}{\sqrt{5}}+b\left(\alpha^{2}+\beta^{2}\right)\right]$$

$$=\frac{1}{2}\left(B_{3n+2}-B_{2}\right)$$

$$=\frac{1}{2}\left[B_{3n+2}-(2b+s)\right]$$

This is completes the proof.

Theorem 5.7 : *For every positive integer* $B_5+B_8+B_{11}+.......+B_{3n+2}=\frac{B_{3n+4}-(4b+3s)}{2}$ (5.11)

Proof: By using Binet's formula, we have

$$B_5+B_8+B_{11}+.......+B_{3n+2}$$

$$=\frac{\alpha^{5}-\beta^{5}}{\sqrt{5}}+b\left(\alpha^{5}-\beta^{5}\right)+\frac{\alpha^{8}-\beta^{8}}{\sqrt{5}}+b\left(\alpha^{8}-\beta^{8}\right)+.....+\frac{\alpha^{3n+2}-\beta^{3n+2}}{\sqrt{5}}+b\left(\alpha^{3n+2}-\beta^{3n+2}\right)$$

$$=\frac{1}{\sqrt{5}}\left[\left(\alpha^{5}+\alpha^{8}+\alpha^{11}.....\alpha^{3n+2}\right)-\left(\beta^{5}+\beta^{8}+\beta^{11}.....\beta^{3n+2}\right)\right]+b\left[\left(\alpha^{5}+\alpha^{8}+\alpha^{11}.....\alpha^{3n+2}\right)-\left(\beta^{5}+\beta^{8}+\beta^{11}.....\beta^{3n+2}\right)\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{\alpha^{3n+5}-\alpha^{5}}{\alpha^{5}-1}\right)-\left(\frac{\beta^{3n+5}-\beta^{5}}{\beta^{5}-1}\right)\right]+b\left[\frac{\alpha^{3n+5}-\alpha^{5}}{\alpha^{5}-1}+\frac{\beta^{3n+5}-\beta^{5}}{\beta^{5}-1}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\frac{\alpha^{3n+4}-\alpha^{4}}{2}-\left(\frac{\beta^{3n+4}-\beta^{4}}{2}\right)\right]+b\left[\frac{\alpha^{3n+4}-\alpha^{4}}{2}+\frac{\beta^{3n+4}-\beta^{4}}{2}\right]$$

$$=\frac{1}{2}\left[\frac{\alpha^{3n+4}-\beta^{3n+4}}{\sqrt{5}}+b\left(\alpha^{3n+4}+\beta^{3n+4}\right)\right]+b\left[\frac{\alpha^{4}+\beta^{4}}{\sqrt{5}}+b\left(\alpha^{4}+\beta^{4}\right)\right]$$

$$=\frac{\left(B_{3n+4}-B_{4}\right)}{2}$$

$$=\frac{\left[B_{3n+4}-(4b+3s)\right]}{2}$$

This is completes the proof.

Theorem 5.8: For positive integer n, prove that

$$B_n^2=(-1)^{n+1}sB_n,\quad n\geq 1 \tag{5.12}$$

This can be derived same as theorem

Theorem 5.9: *For every integer* $n\geq 0$ *, prove that*

$$B_{2n}=F_{2n}+bL_{2n}\quad n\geq 0 \tag{5.13}$$

This can be derived same as theorem

Theorem 5.10: *For every integer* $n\geq 0$ *, prove that*

$$B_n=F_n+bL_n\quad n\geq 0 \tag{5.14}$$

This can be derived same as theorem

6. Connection formulae

In this section, connection formulae of Generalized Fibonacci-Lucas sequence, induction method are presented.

Theorem 6.1: *For positive integer n, prove that*
$2b\ F_{n-1} = B_{n-1} - B_{n-2},\ n \geq 3$ (6.1)

Proof: We shall prove this identity by induction It is easy to show that for $n = 3$
$2bF_{n-1}= 2bF_{3-1}= 2bF_2$
$= 2bF_2$
$=2b .1 = 2b$
$= B_2 - B_1$
Now suppose the identity holds $n = k-1$, $n = k-2$. Then,
$2b\ F_{k-2}= B_{k-2} - B_{k-3}$ (6.2)
$2b\ F_{k-3}= B_{k-3} - B_{k-4}$ (6.3)
On adding equation (6.2) and (6.3), we get
i.e. $2bF_{k-2} + 2bF_{k-3}= (B_{k-2}+ B_{k-3}) - (B_{k-3} + B_{k-4})$
$2b\ (F_{k-2} + F_{k-3}= B_{k-1}- B_{k-2}$
$2b\ F_{k-1}= B_{k-1}- B_{k-2}$
Which is precisely our identity when $n = k$
Hence $2b\ F_{n-1} = B_{n-1} - B_{n-2},\quad n \geq 3$

Theorem 6.2: *For positive integer n, prove that*
$2bL_{n-1} = B_n - B_{n-1},\quad n \geq 2$ (6.4)

Proof: We shall Prove this identity by induction over n. for $n = 2$
$2bL_{n-1}= 2b\ L_{2-1}= 2b\ L_1$
$=2b. 1$
$= 2b$
$= B_2 - B_1$
Now suppose the identity holds for $n = k-1$, $n = k-2$. Then,
$2b\ L_{k-2} = B_{k-1} - B_{k-2}$ (6.5)
$2b\ L_{k-3} = B_{k-2} - B_{k-3}$ (6.6)
Adding equation (6.5) and (6.6), we get
i.e. $2b\ (L_{k-2} + L_{k-3}) = (B_{k-1} + B_{k-2}) - (B_{k-2}+ B_{k-3})$
$2b\ L_{k-1} = B_k - B_{k-1}$
Which is true for $n = k$,
Hence $2b\ L_{n-1} = B_n - B_{n-1},\quad n \geq 2$

Theorem 6.3: *For positive integer n, prove that*
$s\ L_{n-1} = B_{n-1} - F_{n-2},\quad n\geq 2$ (6.7)

Theorem 6.4: *For positive integer n, prove that*
$s\ L_{n-1} = B_n - B_{n-2},\ n \geq 2$ (6.8)

Theorem 6.4: *For positive integer n, prove that*
$B_{n-3} = 2b\ L_{n-2} + F_{n-3},\ n \geq 3$ (6.9)

Theorem 6.5: *For positive integer n, prove that*
$2b\ F_{n-1} = B_{n+1} - 2B_{n-1},\quad n \geq 2$ (6.10)

7. Some determinant identities

Determinants have played a significant part in various areas in mathematics. There are different perspectives on the study of determinants. Problems on determinants of Fibonacci sequence and Lucas sequence are appeared in various issues of Fibonacci Quarterly. Many determinant identities of generalized Fibonacci sequence are discussed in [12]. In this section some determinant identities of generalized Fibonacci-Lucas sequence are derived. Entries of determinants are satisfying the recurrence relation of generalized Fibonacci-Lucas sequence and other sequences.

Theorem 7.1: *Let n be a positive integer. Then*

$$\begin{vmatrix} B_n & F_n & 1 \\ B_{n+1} & F_{n+1} & 1 \\ B_{n+2} & F_{n+2} & 1 \end{vmatrix} = [F_n B_{n+1} - B_n F_{n+1}]$$

Proof: Let $\Delta = \begin{vmatrix} B_n & F_n & 1 \\ B_{n+1} & F_{n+1} & 1 \\ B_{n+2} & F_{n+2} & 1 \end{vmatrix}$ (7.1)

And assume $B_n = a$, $B_{n+1} = b$, $B_{n+2} = a + b$ (7.2)

$F_n = P$, $F_{n+1} = q$, $F_{n+2} = p + q$ (7.3)

Now substituting the value of equation (7.2) & (7.3) in (7.1), we get

$$\Delta = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ $\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$

Applying $R_2 \to R_2 - R_3$ $\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b-(a+b) & q-(p+q) & 0 \\ a+b & p+q & 1 \end{vmatrix}$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ -a & -p & 0 \\ a+b & p+q & 1 \end{vmatrix}$$

$\Delta = [pb - aq]$ (7.4)

Again substituting the values of the equation (7.2) and (7.3) in (7.4), we get $\Delta = [F_n B_{n+1} - B_n F_{n+1}]$

Hence $\begin{vmatrix} B_n & F_n & 1 \\ B_{n+1} & F_{n+1} & 1 \\ B_{n+2} & F_{n+2} & 1 \end{vmatrix} = [F_n B_{n+1} - B_n F_{n+1}]$ similarly we can derive following identities:

Theorem 7.2: *For every integer n ≥ 2, prove that* $\begin{vmatrix} B_n & B_{n+1} & B_{n+2} \\ B_{n+2} & B_n & B_{n+1} \\ B_{n+1} & B_{n+2} & B_n \end{vmatrix} = 2(B_n^3 + B_{n+1}^3)$ (7.5)

Theorem 7.3: *For any integer n ≥0, prove that* $\begin{vmatrix} B_n & L_n & 1 \\ B_{n+1} & L_{n+1} & 1 \\ B_{n+2} & L_{n+2} & 1 \end{vmatrix} = 2(L_n B_{n+1} - B_n L_{n+1})$ (7.6)

Theorem 7.4: For every positive integer n, prove that $\begin{vmatrix} B_n + B_{n+1} & B_{n+1} + B_{n+2} & B_{n+2} + B_n \\ B_{n+2} & B_n & B_{n+1} \\ 1 & 1 & 1 \end{vmatrix} = 0$ (7.7)

Theorem 7.5: For every positive integer n, prove that $\begin{vmatrix} 1+B_n & B_{n+1} & B_{n+2} \\ B_n & 1+B_{n+1} & B_{n+2} \\ B_n & B_{n+1} & 1+B_{n+2} \end{vmatrix} = 1 + B_n + B_{n+1} + B_{n+2}$ (7.8)

The identities from (7.1) to (7.4) can be proved similarly.

8. Conclusion

There are many know identities established for this paper Fibonacci and Lucas sequence. Their paper describes comparable identities of Generalized Fibonacci-Lucas sequence, Fibonacci sequence and Lucas sequence respectively. It is easy to discover new identities simply by varying the pattern of know identities and using inductive reasoning to guess new result.

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