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A new Alzer type Inequality Related to Binomial Function

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Abstract

In this paper, we establish a new Alzer type inequality related to binomial function by using Sitnik methods.

Keywords: binomial function; Alzer type inequality; monotonicity

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1. Introduction

The difference $J_n(x) = e^x - \sum_{k=0}^{n} \frac{x^k}{k!}$ for real x and positive integers n have been studied by many mathematicians. In 1943, P. K. Menon[8] proved the intriguing inequality

$$J_{n-1}(x) J_{n+1}(x) > \frac{1}{2}(J_n(x))^2, \tag{1}$$

which is valid for all positive integers n and for all $x > 0$.
Later, H. Alzer[1] established the sharpened inequality

$$J_{n-1}(x) \cdot J_{n+1}(x) > \frac{n+1}{n+2}(J_n(x))^2, \tag{2}$$

for all $n \in \mathbf{N}$ and $x > 0$, and with the best possible constant $\frac{n+1}{n+2}$.
In 2015, L. Yin and W. -Y. Cui[13] showed a generalization of Alzer inequality related to exponential function, and generalized it for the remainder of Maclaurin series.
Recently, S. M. Sitnik formulated some conjectures on monotonicity of ratios for exponential series remainders. They are equivalent to conjectures on monotonicity of a ratio of Kummer hypergeometric function , see [9] and [10]. Afterwards, K. Mehrez and S. M. Sitnik proved their conjectures in [7].
Actually, the Turán type inequalities have a more extensive literature and recently the results have been applied in problems arising from many fields such as information theory, economic theory and biophysics.For more about this subject the readers refer to [2, 3, 4, 5, 8, 6, 9, 11, 12] and the references therein.
In this paper,we consider the following binomial function

$$I_n(x) = \frac{1}{\sqrt[p]{1+x^p}} - \sum_{k=0}^{n} (-1)^k \frac{p(p+1)\cdots(p+k-1)x^{kp}}{k!}$$

$$= \sum_{k=n+1}^{\infty} (-1)^k \frac{p(p+1)\cdots(p+k-1)x^{kp}}{k!}.$$

where

$$\frac{1}{\sqrt[p]{1+x^p}} = 1 - px^p + \frac{p(p+1)}{2!}x^{2p}$$

$$+ \cdots + (-1)^n \frac{p(p+1)\cdots(p+n-1)}{n!}x^{np} + \cdots.$$

The main purpose of this note is to find the greatest value $C_{n,k}$, such that

$$I_{n-k}(x)I_{n+k}(x) > C_{n,p,k}(I_n(x))^2 \tag{3}$$

is valid for every positive x and $n, k \in \mathbf{N}$.
We established the sharpened inequality

$$\frac{I_{n-1}(z)I_{n+1}(z)}{I_n^2(z)} > \frac{(p+n+1)(n+1)}{(p+n)(n+2)}, \tag{4}$$

and

$$\frac{I_{n-k}(z)I_{n+k}(z)}{I_n^2(z)} > \frac{(p+n+1)\cdots(p+n+k)(n-k+2)\cdots(n+1)}{(p+n-k+1)\cdots(p+n)(n+2)\cdots(n+k+1)}. \tag{5}$$

2. lemmas

Lemma 1 ([7]). Let $\{a_n\}$ and $\{b_n\}, (n=0,1,2,\cdots)$ be real numbers such that $b_n > 0$ and $\{\frac{a_n}{b_n}\}_{n\geq 0}$ is increasing(decreasing), then $\{\frac{a_0+a_1+\cdots+a_n}{b_0+b_1+\cdots+b_n}\}$ is increasing(decreasing).

Lemma 2 ([7]). Let $\{a_n\}$ and $\{b_n\}, (n=0,1,2,\cdots)$ be real numbers and let the power series $A(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ be convergent if $|x| < r$. If $b_n > 0, (n=0,1,2,\cdots)$ and the sequence $\{\frac{a_n}{b_n}\}_{n\geq 0}$ is (strictly)increasing(decreasing), then the function $\frac{A(x)}{B(x)}$ is also (strictly) increasing(decreasing) on $[0,r)$.

3. main results

Theorem 1. *For every* $n, k \in \mathbf{N}, p \geq 2$, *and* $0 < z < 1$. *The function*

$$E(n,p,k,z) = \frac{I_{n-k}(z)I_{n+k}(z)}{I_n^2(z)} \tag{6}$$

is strictly increasing on $(0,\infty)$. *As a result, we have the following Turán type inequalities*

$$\frac{I_{n-k}(z)I_{n+k}(z)}{I_n^2(z)} > \frac{(p+n+1)\cdots(p+n+k)(n-k+2)\cdots(n+1)}{(p+n-k+1)\cdots(p+n)(n+2)\cdots(n+k+1)} \tag{7}$$

where the constant $\frac{(p+n+1)\cdots(p+n+k)(n-k+2)\cdots(n+1)}{(p+n-k+1)\cdots(p+n)(n+2)\cdots(n+k+1)}$ *is best possible.*

Proof. simple computation yields

$$E(n,p,k,z) = \frac{I_{n-k}(z)I_{n+k}(z)}{I_n^2(z)}$$

$$= \frac{\sum\limits_{m=0}^{\infty}\sum\limits_{j=0}^{m} \frac{(p(p+1)\cdots(p+n-k+j))(p\cdots(p+n+k+m-j))}{(n-k+1+j)!(n+k+1+m-j)!}(-1)^{2n+2+m}z^{(2n+2+m)p}}{\sum\limits_{m=0}^{\infty}\sum\limits_{j=0}^{m} \frac{(p\cdots(p+n+j))(p(p+1)\cdots(p+n+m-j))}{(n+1+j)!(n+1+m-j)!}(-1)^{2n+2+m}z^{(2n+2+m)p}}$$

Define sequences $\{A_{n,p,k,j}\}, \{B_{n,p,k,j}\}$ and $\{C_{n,p,k,j}\}$ by

$$A_{n,p,k,j} = \frac{(p\cdots(p+n-k+j))(p\cdots(p+n+k+m-j))}{(n-k+1+j)!(n+k+1+m-j)!}(-1)^{2n+2+m},$$

$$B_{n,p,k,j} = \frac{(p\cdots(p+n+j))(p\cdots(p+n+m-j))}{(n+1+j)!(n+1+m-j)!}(-1)^{2n+2+m},$$

and

$$C_{n,p,k,j} = \frac{A_{n,p,k,j}}{B_{n,p,k,j}}$$

$$= \frac{\frac{(p(p+1)\cdots(p+n-k+j))(p(p+1)\cdots(p+n+k+m-j))}{(n-k+1+j)!(n+k+1+m-j)!}(-1)^{2n+2+m}}{\frac{(p(p+1)\cdots(p+n+j))(p(p+1)\cdots(p+n+m-j))}{(n+1+j)!(n+1+m-j)!}(-1)^{2n+2+m}}$$

$$= \frac{(p(p+1)\cdots(p+n-k+j))}{(p(p+1)\cdots(p+n+j))}$$

$$\frac{(p(p+1)\cdots(p+n+k+m-j))(n+1+j)!(n+1+m-j)!}{(p(p+1)\cdots(p+n+m-j))(n-k+1+j)!(n+k+1+m-j)!}.$$

$$= \frac{(n+j+1-k+1)\cdots(n+j+1)}{(n+1+m-j+1)\cdots(n+1+m-j+k)}$$

$$\frac{(p+n+m-j+1)\cdots(p+n+m-j+k)}{(p+n+j-k+1)\cdots(p+n+j)}.$$

Further computation results in

$$\frac{C_{n,p,k,j+1}}{C_{n,p,k,j}} = \frac{(p+n+m-j)(p+n+j-k+1)}{(p+n+j+1)(p+n+m-j+k)}$$

$$\cdot\frac{(n+2+j)(n+m-j+k+1)}{(n+j-k+2)(n+m-j+1)}.$$

$$= \frac{((n+m-j)^2+(p+k+1)(n+m-j)+p(k+1))}{((n+m-j)^2+(p+k+1)(n+m-j)+p+k)}$$

$$\cdot\frac{((n+j)^2+(p-k+3)(n+j)+2(p-k+1))}{((n+j)^2+(p-k+3)(n+j)+(p+1)(2-k))}. \tag{8}$$

When $p \geq 2$, then $p(k+1) > p+k$, and $2(p-k+1)-(p+1)(2-k) = (p-2)k \geq 0$, so we have $\frac{C_{n,p,k,j+1}}{C_{n,p,k,j}} > 1$. This implies that the sequence $C_{n,p,k,j}$ is strictly increasing to j. By using Lemma 2.1 and Lemma 2.2, we easily obtain the function $E(n,p,k,z)$ is strictly increasing on $(0,\infty)$.

Finally, from limit identity

$$\lim_{z\to 0^+}\frac{I_{n-k}(z)I_{n+k}(z)}{I_n^2(z)} = \frac{(p+n+1)\cdots(p+n+k)(n-k+2)\cdots(n+1)}{(p+n-k+1)\cdots(p+n)(n+2)\cdots(n+k+1)}, \tag{9}$$

we complete the proof. $\square$

In particular, taking $k = 1$ in the Theorem 3.1, we easily obtain the following Corollary 3.1.

Corollary 1. For every $n \in \mathbf{N}$, $p \geq 2$, and $0 < z < 1$. The function

$$E(n,p,z) = \frac{I_{n-1}(z)I_{n+1}(z)}{I_n^2(z)} \tag{10}$$

is strictly increasing on $(0,\infty)$. As a result, we have the following Turán type inequalities

$$\frac{I_{n-1}(z)I_{n+1}(z)}{I_n^2(z)} > \frac{(p+n+1)(n+1)}{(p+n)(n+2)} \tag{11}$$

where the constant $\frac{(p+n+1)(n+1)}{(p+n)(n+2)}$ is best possible.

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