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Research Paper

Some spaces of sequences of interval numbers defined by a modulus function

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Abstract

The main purpose of the present paper is to introduce $\overline{c}_o\left(f,p,s\right)$, $\overline{c}\left(f,p,s\right)$ $\overline{l}_\infty\left(f,p,s\right)$ and $\overline{l}_p\left(f,p,s\right)$ of sequences of interval numbers defined by a modulus function. Furthermore, some inclusion theorems related to these spaces are given.

Keywords: *Complete space, interval number, modulus function.*

1. Introduction

Interval arithmetic was first suggested by Dwyer [8] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [11] in 1959 and Moore and Yang [13] 1962. Furthermore, Moore and others [9], [10], [11] and [14] have developed applications to differential equations.

Chiao in [6] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [7] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently, Esi in [1], [2], [3], [4] and [5] defined and studied different properties of interval numbers.

We denote the set of all real valued closed intervals by $\mathrm{I\mathbb{R}}$. Any elements of $\mathrm{I\mathbb{R}}$ is called interval number and denoted by $\overline{A}=\left[x_l,x_r\right]$. Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{A}_1,\overline{A}_2\in\mathrm{I\mathbb{R}}$, we have $\overline{A}_1=\overline{A}_2\Leftrightarrow x_{1_l}=x_{2_l}, x_{1_r}=x_{2_r}$. $\overline{A}_1+\overline{A}_2=\left\{x\in\mathbb{R}: x_{1_l}+x_{2_l}\leq x\leq x_{1_r}+x_{2_r}\right\}$,and if $\alpha\geq 0$, then $\alpha\overline{A}=\left\{x\in\mathbb{R}:\ \alpha x_{1_l}\leq x\leq\alpha x_{1_r}\right\}$ and if $\alpha<0$, then $\alpha\overline{A}=\left\{x\in\mathbb{R}:\ \alpha x_{1_r}\leq x\leq\alpha x_{1_l}\right\}$,

$$\overline{A}_1.\overline{A}_2=\left\{x\in\mathbb{R}:\min\left\{x_{1_l}.x_{2_l},x_{1_l}.x_{2_r},x_{1_r}.x_{2_l},x_{1_r}.x_{2_r}\right\}\leq x\leq\max\left\{x_{1_l}.x_{2_l},x_{1_l}.x_{2_r},x_{1_r}.x_{2_l},x_{1_r}.x_{2_r}\right\}\right\}.$$

The set of all interval numbers $\mathrm{I\mathbb{R}}$ is a complete metric space defined by

$$\overline{d}\left(\overline{A}_1,\overline{A}_2\right)=\max\left\{\left|x_{1_l}-x_{2_l}\right|,\left|x_{1_r}-x_{2_r}\right|\right\}$$ [12].

In the special case $\overline{A}_1=[a,a]$ and $\overline{A}_2=[b,b]$, we obtain usual metric of $\mathbb{R}$. Let us define transformation $f:\mathbb{N}\rightarrow\mathbb{R}$ by $k\rightarrow f(k)=\overline{A}$, $\overline{A}=\left(\overline{A}_k\right)$. Then $\overline{A}=\left(\overline{A}_k\right)$ is called sequence of interval numbers. The $\overline{A}_k$ is called k^{th} term of sequence $\overline{A}=\left(\overline{A}_k\right)$. $\overline{w}$ denotes the set of all interval numbers with real terms and the algebraic properties of $\overline{w}$ can be found in [14].

Now we give the definition of convergence of interval numbers:

Definition 1.1 *([6]) A sequence $\overline{A}=\left(\overline{A}_k\right)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon>0$ there exists a positive integer k_o such that $\overline{d}\left(\overline{A}_k,\overline{A}_o\right)<\varepsilon$ for all $k\geq k_o$ and we denote it by* $\lim_k \overline{A}_k=\overline{A}_o$.

Thus, $\lim_k \overline{A}_k=\overline{A}_o \Leftrightarrow \lim_k A_{k_l}=A_{o_l}$ and $\lim_k A_{k_r}=A_{o_r}$.

We recall that modulus function is a function $f:[0,\infty)\to[0,\infty)$ such that

(a) $f(x)=0$ if and only if $x=0$,

(b) $f(x+y)\leq f(x)+f(y)$ for all $x,y\geq 0$,

(c) f is increasing,

(d) f is continuous from the right at zero.

It follows from (a) and (d) that f must be continuous everywhere on $[0,\infty)$.

Let $p=(p_k)$ be a bounded sequence of strictly positive real numbers. If $H=sup_k p_k$, then for any complex numbers a_k and b_k

$$|a_k+b_k|^{p_k}\leq C\left(|a_k|^{p_k}+|b_k|^{p_k}\right) \tag{1}$$

where $C=\max(1,2^{H-1})$.

Definition 1.2 *A set of X sequences of interval numbers is said to be solid (or normal) if $\left(\overline{B}_k\right)\in X$ whenever $\overline{d}\left(\overline{B}_k,\overline{0}\right)\leq\overline{d}\left(\overline{A}_k,\overline{0}\right)$ for all $k\in\mathbb{N}$, for some $\left(\overline{A}_k\right)\in X$.*

In this paper, we essentially deal with the metric spaces $\overline{c}_o(f,p,s)$, $\overline{c}(f,p,s)$, $\overline{l}_\infty(f,p,s)$ and $\overline{l}_p(f,p,s)$ of sequences of interval numbers defined by a modulus function which are generalization of the metric spaces $\overline{c}_o$, $\overline{c}$, $\overline{l}_\infty$ and $\overline{l}_p$ of sequences of interval numbers. We state and prove some topological and inclusion theorems related to those sets.

2. Main results

Let f be a modulus function and $s\geq 0$ be a real number and $p=(p_k)$ be a sequence of strictly positive real numbers. We introduce the sets of sequences of interval numbers as follows:

$$\overline{c}_o(f,p,s)=\left\{\overline{A}=\left(\overline{A}_k\right):\ \lim_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k}=0\right\},$$

$$\overline{c}(f,p,s)=\left\{\overline{A}=\left(\overline{A}_k\right):\ \lim_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}=0\right\},$$

$$\overline{l}_\infty(f,p,s)=\left\{\overline{A}=\left(\overline{A}_k\right):\ \sup_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k}<\infty\right\}$$

and

$$\overline{l}_p(f,p,s)=\left\{\overline{A}=\left(\overline{A}_k\right):\ \sum_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k}<\infty\right\}.$$

Now, we may begin with the following theorem.

Theorem 2.1 *The sets $\overline{c}_o(f,p,s)$, $\overline{c}(f,p,s)$, $\overline{l}_\infty(f,p,s)$ and $\overline{l}_p(f,p,s)$ of sequences of interval numbers are closed under the coordinatewise addition and scalar multiplication.*

Proof. It is easy, so we omit the detail. ∎

Theorem 2.2 *The sets $\overline{c}_o(f,p,s)$, $\overline{c}(f,p,s)$, $\overline{l}_\infty(f,p,s)$ and $\overline{l}_p(f,p,s)$ of sequences of interval numbers are complete metric spaces with respect to the metrics*

$$\overline{d}_\infty(A,B)=\sup_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{\frac{p_k}{M}}$$

and

$$\overline{d}_p(A,B)=\left\{\sum_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{B}_k\right)\right)\right]^{p_k}\right\}^{\frac{1}{M}}$$

respectively, where $\overline{A}=\left(\overline{A}_k\right)$ *and* $\overline{B}=\left(\overline{B}_k\right)$ *are elements of the sets* $\overline{c}_o(f,p,s)$, $\overline{c}_o(f,p,s)$, $\overline{l}_\infty(f,p,s)$ *and* $\overline{l}_p(f,p,s)$ *and* $M=\max\left(1,\sup_k p_k=H\right)$

Proof. We consider only the space $\overline{c}_o(f,p,s)$, since the proof is similar for the spaces $\overline{c}(f,p,s)$, $\overline{l}_\infty(f,p,s)$ and $\overline{l}_p(f,p,s)$. One can easily establish that $\overline{d}_\infty$ defines a metric on $\overline{c}_o(f,p,s)$ which is a routine verification, so we omit it. It remains to prove the completeness of the space $\overline{c}_o(f,p,s)$. Let $\left(\overline{A}^i\right)$ be any Cauchy sequence in the space $\overline{c}_o(f,p,s)$, where $\overline{A}^i=\left(\overline{A}_o^{(i)},\overline{A}_1^{(i)},\overline{A}_2^{(i)},...\right)$. Then, for a given $\varepsilon>0$ there exists a positive integer $n_o(\varepsilon)$ such that

$$\overline{d}_\infty\left(\overline{A}^i,\overline{A}^j\right)=\sup_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k^{(i)},\overline{A}_k^{(j)}\right)\right)\right]^{\frac{p_k}{M}}<\varepsilon \tag{2}$$

for all $i,j>n_o(\varepsilon)$. We obtain for each fixed $k\in\mathbb{N}$ from (2) that

$$k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k^{(i)},\overline{A}_k^{(j)}\right)\right)\right]^{\frac{p_k}{M}}<\varepsilon \tag{3}$$

for all $i,j>n_o(\varepsilon)$. (3) means that

$$\lim_{i,j\to\infty}k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k^{(i)},\overline{A}_k^{(j)}\right)\right)\right]^{\frac{p_k}{M}}=0. \tag{4}$$

Since $k^{-s}\neq 0$ for all $k\in\mathbb{N}$ and f is continuous, we have from (4) that

$$f\left[\lim_{i,j\to\infty}\left(\overline{d}\left(\overline{A}_k^{(i)},\overline{A}_k^{(j)}\right)\right)\right]=0. \tag{5}$$

Therefore, since f is a modulus function one can derive by (5) that

$$\lim_{i,j\to\infty}\overline{d}\left(\overline{A}_k^{(i)},\overline{A}_k^{(j)}\right)=0 \tag{6}$$

which means that $\left(\overline{A}_k^{(i)}\right)$ is a Cauchy sequence in $\mathbb{R}$ for every fixed $k\in\mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $\overline{A}_k^{(i)}\to\overline{A}_k$ as $i\to\infty$. Using these infinitely many limits, we defined the interval sequence $\left(\overline{A}_k\right)=\left(\overline{A}_o,\overline{A}_1,\overline{A}_2,...\right)$. Let us pass to limit firstly as $j\to\infty$ and nextly taking supremum over $k\in\mathbb{N}$ in (3) we obtain $\overline{d}_\infty\left(\overline{A}^i,\overline{A}_k\right)\leq\varepsilon$. Since $\left(\overline{A}_k^{(i)}\right)\in\overline{c}_o(f,p,s)$ for each $i\in\mathbb{N}$, there exists $k_o\in\mathbb{N}$ such that

$$k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k^{(i)},\overline{0}\right)\right)\right]^{p_k}<\varepsilon$$

for every $k\geq k_o(\varepsilon)$ and for each fixed $i\in\mathbb{N}$. Therefore, since

$$k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k}\leq Ck^{-s}\left[f\left(\overline{d}\left(\overline{A}_k^{(i)},\overline{A}_k\right)\right)\right]^{p_k}+Ck^{-s}\left[f\left(\overline{d}\left(\overline{A}_k^{(i)},\overline{0}\right)\right)\right]^{p_k}$$

hold by triangle inequality for all $i,k\in\mathbb{N}$, where $C=\max(1,2^{H-1})$. Now for all $k\geq k_o(\varepsilon)$, we have

$$k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k}\leq 2\varepsilon.$$

This shows that $\left(\overline{A}_k\right)\in\overline{c}_o(f,p,s)$. Since $\left(\overline{A}_k^{(i)}\right)$ was an arbitrary Cauchy sequence, the the space $\overline{c}_o(f,p,s)$ is complete. ■

Theorem 2.3 *The spaces* $\overline{c}_o(f,p,s)$, $\overline{l}_\infty(f,p,s)$ *and* $\overline{l}_p(f,p,s)$ *are solid.*

Proof. Let $\overline{X}(f,p,s)$ denotes the anyone of the spaces $\overline{c}_o(f,p,s)$, $\overline{l}_\infty(f,p,s)$ and $\overline{l}_p(f,p,s)$. Suppose that

$$\overline{d}\left(\overline{B}_k,\overline{0}\right) \leq \overline{d}\left(\overline{A}_k,\overline{0}\right) \tag{7}$$

holds for some $\left(\overline{A}_k\right) \in \overline{X}(f,p,s)$. Since the modulus function is increasing, one can easily see by (7) that

$$\lim_k k^{-s}\left[f\left(\overline{d}\left(\overline{B}_k,\overline{0}\right)\right)\right]^{p_k} \leq \lim_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k},$$

$$\sup_k k^{-s}\left[f\left(\overline{d}\left(\overline{B}_k,\overline{0}\right)\right)\right]^{p_k} \leq \sup_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k}$$

and

$$\sum_k k^{-s}\left[f\left(\overline{d}\left(\overline{B}_k,\overline{0}\right)\right)\right]^{p_k} \leq \sum_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k}.$$

The above inequalities yield the desired that $\left(\overline{B}_k\right) \in \overline{X}(f,p,s)$. ∎

Theorem 2.4 *Let* $\inf_k p_k = h > 0$. *Then*
a) $\left(\overline{A}_k\right) \in \overline{c}$ *implies* $\left(\overline{A}_k\right) \in \overline{c}(f,p,s)$,
b) $\left(\overline{A}_k\right) \in \overline{c}(p,s)$ *implies* $\left(\overline{A}_k\right) \in \overline{c}(f,p,s)$,
c) $\beta = \lim_t \frac{f(t)}{t} > 0$ *implies* $\overline{c}(p,s) = \overline{c}(f,p,s)$.

Proof. a) Suppose that $\left(\overline{A}_k\right) \in \overline{c}$. Then $\lim_k \overline{d}\left(\overline{A}_k,\overline{A}_o\right) = 0$. As f is modulus function, then

$$\lim_k f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right) = f\left[\lim_k\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right] = f(0) = 0.$$

As $\inf_k p_k = h > 0$, then $\lim_k\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^h = 0$. So, for $0 < \varepsilon < 1$, $\exists k_o$ such that for all $k > k_o$ $\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^h < \varepsilon < 1$, an as $p_k \geq h$ for all k,

$$\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k} \leq \left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^h < \varepsilon < 1,$$

then we obtain

$$\lim_k\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k} = 0.$$

As (k^{-s}) is bounded, we can write

$$\lim_k k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k} = 0.$$

Therefore $\left(\overline{A}_k\right) \in \overline{c}(f,p,s)$.

b) Let $\left(\overline{A}_k\right) \in \overline{c}(p,s)$, then $\lim_k k^{-s}\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)^{p_k} = 0$. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$, such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Now we write

$$I_\imath = \left\{k \in \mathbb{N} : \overline{d}\left(\overline{A}_k,\overline{A}_o\right) \leq \delta\ \right\}$$

and

$$I_2 = \left\{k \in \mathbb{N} : \overline{d}\left(\overline{A}_k,\overline{A}_o\right) > \delta\ \right\}.$$

For $\overline{d}\left(\overline{A}_k,\overline{A}_o\right) > \delta$

$$\overline{d}\left(\overline{A}_k,\overline{A}_o\right) < \overline{d}\left(\overline{A}_k,\overline{A}_o\right)\delta^{-1} < 1 + \left[\left|\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right|\right]$$

where $k \in I_2$ and $[|t|]$ denotes the integer of t. By using properties of modulus function, for $\overline{d}\left(\overline{A}_k,\overline{A}_o\right) > \delta$, we have

$$f\left[\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right] < 1 + \left[\left|\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right|\right] f(1) \leq 2f(1)\,\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\delta^{-1}.$$

For $\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\le\delta$, $f\left[\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right]<\varepsilon$, where $k\in I_1$. Hence

$$k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}=k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}\rfloor_{k\in I_1}+k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}\rfloor_{k\in I_2}$$

$$\le k^{-s}\varepsilon^H+\left[2f\left(1\right)\delta^{-1}\right]^H k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}\to 0 \text{ as } k\to\infty.$$

Then we obtain $\left(\overline{A}_k\right)\in\overline{c}\left(f,p,s\right)$.

c) In (b), it was shown that $\overline{c}\left(p,s\right)\subset\overline{c}\left(f,p,s\right)$. We must show that $\overline{c}\left(f,p,s\right)\subset\overline{c}\left(p,s\right)$. For any modulus function, the existence of positive limit given by β in Maddox[16, Proposition 1]. Now, $\beta>0$ and let $\left(\overline{A}_k\right)\in\overline{c}\left(f,p,s\right)$. As $\beta>0$ for every $t>0$, we write $f(t)\ge\beta t$. From this inequality, it is easy seen that $\left(\overline{A}_k\right)\in\overline{c}\left(p,s\right)$. ∎

Theorem 2.5 *Let f and g be two modulus functions and $s,s_1,s_2\ge 0$. Then*

a) $\overline{c}\left(f,p,s\right)\cap\overline{c}\left(g,p,s\right)\subset\overline{c}\left(f+g,p,s\right)$,

b) $s_1\le s_2$ *implies* $\overline{c}\left(f,p,s_1\right)\subset\overline{c}\left(f,p,s_2\right)$.

Proof. a) Let $\left(\overline{A}_k\right)\in\overline{c}\left(f,p,s\right)\cap\overline{c}\left(g,p,s\right)$. From (1), we have

$$\left[\left(f+g\right)\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}=\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)+g\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}$$

$$\le C\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}+C\left[g\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}.$$

As (k^{-s}) is bounded, we can write

$$k^{-s}\left[\left(f+g\right)\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}$$

$$\le Ck^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}+Ck^{-s}\left[g\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}.$$

Hence we obtain $\left(\overline{A}_k\right)\in\overline{c}\left(f+g,p,s\right)$.

b) Let $s_1\le s_2$. Then $k^{-s_2}\le k^{-s_1}$ for all $k\in\mathbb{N}$. Hence

$$k^{-s_2}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}\le k^{-s_1}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{A}_o\right)\right)\right]^{p_k}.$$

This inequality implies that $\overline{c}\left(f,p,s_1\right)\subset\overline{c}\left(f,p,s_2\right)$. ∎

Theorem 2.6 *Let f be a modulus function, then*

a) $\overline{l}_\infty\subset\overline{l}_\infty\left(f,p,s\right)$,

b) *If f is bounded then* $\overline{l}_\infty\left(f,p,s\right)=\overline{w}$.

Proof. a) Let $\left(\overline{A}_k\right)\in\overline{l}_\infty$. Then there exists a positive integer M such that $\overline{d}\left(\overline{A}_k,\overline{0}\right)\le M$. Since f is bounded then $f\left[\overline{d}\left(\overline{A}_k,\overline{0}\right)\right]$ is also bounded. Hence

$$k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k}\le k^{-s}\left[Mf(1)\right]^{p_k}\le k^{-s}\left[Mf(1)\right]^H<\infty.$$

Therefore $\left(\overline{A}_k\right)\in\overline{l}_\infty\left(f,p,s\right)$.

b) If f is bounded, then for any $\left(\overline{A}_k\right)\in\overline{w}$,

$$k^{-s}\left[f\left(\overline{d}\left(\overline{A}_k,\overline{0}\right)\right)\right]^{p_k}\le k^{-s}L^{p_k}\le k^{-s}L^H<\infty.$$

Hence $\overline{l}_\infty\left(f,p,s\right)=\overline{w}$. ∎

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