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Short Communication

Almost convergence of triple sequences

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Abstract

In this paper we introduce and study the concepts of almost convergence and almost Cauchy for triple sequences.We show that the set of almost convergent triple sequences of 0's and 1's is of the first category and also almost every triple sequence of 0's and 1's is not almost convergent.

Keywords: *almost convergence, P-convergent, triple sequence.*

1. Introduction

Before we enter the motivation for this paper and the presentation of the main results we give some preliminaries.

The concept of almost convergence of sequences of real numbers $x = (x_n)$ was introduced and firstly studied by Lorentz [2]. A sequence $x = (x_n)$ almost converges to the number L if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \frac{1}{k} \sum_{i=0}^{k-1} x_{n+i} - L \right| < \varepsilon, \text{ for all } k > N \text{ and for all } n \in \mathbb{N}.$$

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{n,m})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{n,m} - L| < \varepsilon$ whenever $n, m > N$, [4]. We shall write more briefly as "*P-convergent*".

Later, Moricz and Rhoades [3] have expanded the definition of almost convergence to double sequences as follows: A double sequence $x = (x_{n,m})$ of real numbers is called almost P-convergent to a limit L if,

$$P - \lim_{p,q\to\infty} \frac{1}{pq} \sum_{n=s}^{s+p-1} \sum_{m=t}^{t+q-1} |x_{n,m} - L| = 0$$

that is, the average value of $(x_{n,m})$ taken over any rectangle

$$\{(n, m) : \ s \le n \le s + p - 1, t \le m \le t + q - 1\}$$

tends to L as both p and q tend to infinity, and this P-convergence is uniform in s and t.

In 2007, Cunjalo [1] studied the definition of almost Cauchy and showed that the set of almost convergent double sequences of 0's and 1's is of the first category and almost every double sequences of 0's and 1's is not almost convergent.

2. Definitions and Results

Let X denote the set of all triple sequences of 0's and 1's, namely

$$X = \{x = (x_{n,m,l}) : x_{n,m,l} \in \{0,1\}\,,\ n,m,l \in \mathbb{N}\}\,.$$

Let B be the smallest $\sigma-$algebra of subsets of the set X which contains all sets of the form:

$$\{x = (x_{n,m,l}) \in X :\ x_{n,m,l} = a_1, x_{n_1,m_1,l_1} = a_2, ..., x_{n_k,m_k,l_k} = a_k\}\,, a_1, a_2, ..., a_k \in \{0,1\}\,, k \in \mathbb{N}.$$

There exists the unique Lebesgue measure R on the set X, such that

$$R\left(\{x = (x_{n,m,l}) \in X : x_{n,m,l} = a_1, x_{n_1,m_1,l_1} = a_2, ..., x_{n_k,m_k,l_k} = a_k\}\right) = \frac{1}{2^k}$$

where $a_1, a_2, ..., a_k \in \{0,1\}\,, k \in \mathbb{N}$.

The set X equipped with the metric $d : X \times X \to \mathbb{R}^+$,

$$d\left(x,y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{|x_{nml} - y_{nml}|}{2^{n+m+l}},$$

is a complete metric space. Therefore, X is of the second category. The aim of this paper is to generalize Cunjalo's results for triple sequences.

Definition 2.1 *The triple sequence $x = (x_{n,m,l})$ almost converges to L, if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that*

$$\left|\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1} x_{n+i,m+j,l+k} - L\right| < \varepsilon$$

for all $p,q,r > N$ and all $(n,m,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

Definition 2.2 *The triple sequence $x = (x_{n,m,l})$ is almost Cauchy, if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that*

$$\left|\frac{1}{p_1q_1r_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\sum_{k=0}^{r_1-1} x_{n_1+i,m_1+j,l_1+k} - \frac{1}{p_2q_2r_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\sum_{k=0}^{r_2-1} x_{n_2+i,m_2+j,l_2+k}\right| < \varepsilon$$

for all $p_1, p_2, q_1, q_2, r_1, r_2 > N$ and all $(n_1,m_1,l_1)\,,(n_2,m_2,l_2) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

Lemma 2.3 *The triple sequence $x = (x_{n,m,l})$ is almost convergent if and only if it is almost Cauchy.*

Proof. Suppose that the triple sequence $x = (x_{n,m,l})$ is almost convergent to L. Then, for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\left|\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1} x_{n+i,m+j,l+k} - L\right| < \varepsilon$$

for all $p,q,r > N$ and all $(n,m,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$. Therefore

$$\left|\frac{1}{p_1q_1r_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\sum_{k=0}^{r_1-1} x_{n_1+i,m_1+j,l_1+k} - \frac{1}{p_2q_2r_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\sum_{k=0}^{r_2-1} x_{n_2+i,m_2+j,l_2+k}\right|$$

$$\leq \left|\frac{1}{p_1q_1r_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\sum_{k=0}^{r_1-1} x_{n_1+i,m_1+j,l_1+k} - L\right| + \left|\frac{1}{p_2q_2r_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\sum_{k=0}^{r_2-1} x_{n_2+i,m_2+j,l_2+k} - L\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $p_1,p_2,q_1,q_2,r_1,r_2>N$ and all $(n_1,m_1,l_1),(n_2,m_2,l_2)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. Hence, the triple sequence $x=(x_{n,m,l})$ is almost Cauchy.

Now, suppose that $x=(x_{n,m,l})$ is almost Cauchy. Then, for every $\varepsilon>0$, $\exists N\in\mathbb{N}$ such that

$$\left|\frac{1}{p_1q_1r_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\sum_{k=0}^{r_1-1}x_{n_1+i,m_1+j,l_1+k}-\frac{1}{p_2q_2r_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\sum_{k=0}^{r_2-1}x_{n_2+i,m_2+j,l_2+k}\right|<\frac{\varepsilon}{2} \tag{1}$$

for all $p_1,p_2,q_1,q_2,r_1,r_2>N$ and all $(n_1,m_1,l_1),(n_2,m_2,l_2)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. Taking $n_1=n_2=n_o, m_1=m_2=m_o$ and $l_1=l_2=l_o$ in relation (1), we obtain that

$$\left(\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1}x_{n_\imath+i,m_\imath+j,l_\imath+k}\right)_{p,q,r=1}^{\infty}$$

is a Cauchy sequence in $\mathbb{C}$, the set of complex numbers, and therefore it is convergent since $\mathbb{C}$ is complete. Let

$$P-\lim_{p,q,r\to\infty}\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1}x_{n_\imath+i,m_\imath+j,l_\imath+k}=L.$$

Then, for every $\varepsilon>0$, $\exists N_1\in\mathbb{N}$ such that

$$\left|\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1}x_{n_\imath+i,m_\imath+j,l_\imath+k}-L\right|<\frac{\varepsilon}{2}$$

for all $p,q,r>N_1$. It follows that

$$\left|\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1}x_{n+i,m+j,l+k}-L\right|$$

$$\leq\left|\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1}x_{n+i,m+j,l+k}-\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1}x_{n_\imath+i,m_\imath+j,l_\imath+k}\right|$$

$$+\left|\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1}x_{n_\imath+i,m_\imath+j,l_\imath+k}-L\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

for all $p,q,r>\max(N,N_1)$ and for all $(n,m,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. So, the triple sequence $x=(x_{n,m,l})$ is almost convergent to L. This completes the proof. ■

Theorem 2.4 *The set*

$$V=\{x\in X : x=(x_{n,m,l})\ \textit{ is almost convergent }\}$$

is of the first category.

Proof. Due to Lemma 2.3, the set V can be represented in the form

$$V=\cap_{s=1}^{\infty}\cup_{t=1}^{\infty}\cap^{\infty}_{\substack{p_1,p_2=t\\ q_1,q_2=t\\ r_1,r_2=t}}\cap^{\infty}_{\substack{n_1,n_2=1\\ m_1,m_2=1\\ l_1,l_2=1}}\left\{x\in X:\left|D^{p_1,q_1,r_1}_{n_1,m_1,l_1}(x)-D^{p_2,q_2,r_2}_{n_2,m_2,l_2}(x)\right|<\frac{1}{s}\right\}$$

where

$$D^{p,q,r}_{n,m,l}(x)=\frac{1}{pqr}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1}x_{n+i,m+j,l+k}.$$

Now, let

$$T = \{x \in X : \; x_{n,m,l} = 1 \text{ for finitely many } \; (n,m,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\}.$$

The set T is everywhere dense in X. For arbitrary $y \in T$, $\exists n_1, m_1, l_1, p_1, q_1, r_1 \in \mathbb{N}$ and $p_1, q_1, r_1 > t$ such that

$$\{(n,m,l) : y_{n,m,l} = 1\} \subset \{(n,m,l) : n_1 \leq n < n_1 + p_1, m_1 \leq m < m_1 + q_1, l_1 \leq l < l_1 + r_1\},$$

$$D^{p_1,q_1,r_1}_{n_1,m_1,l_1}(y) \leq \frac{1}{3s_o}.$$

Let $z = (z_{n,m,l}) \in T$ such that

$$z_{nml} = \{ \begin{matrix} y_{nml}, n_1 \leq n < n_1 + p_1, m_1 \leq m < m_1 + q_1, l_1 \leq l < l_1 + r_1 \\ 1, \quad n_2 \leq n < n_2 + p_2, m_2 \leq m < m_2 + q_2, l_2 \leq l < l_2 + r_2 \end{matrix}$$

where $n_2 > n_1 + p_1, m_2 > m_1 + q_1, l_2 > l_1 + r_1$. Then

$$\left| D^{p_1,q_1,r_1}_{n_1,m_1,l_1}(z) - D^{p_2,q_2,r_2}_{n_2,m_2,l_2}(z) \right| > \frac{1}{s_o}.$$

Now, let $z' = \left(z'_{n,m,l}\right) \in X$ with the property $z'_{n,m,l} = z_{n,m,l}$ on the set

$$\{(n,m,l) : \; n_1 \leq n < n_1 + p_1, m_1 \leq m < m_1 + q_1, l_1 \leq l < l_1 + r_1\}$$

$$\cup \{(n,m,l) : n_2 \leq n < n_2 + p_2, m_2 \leq m < m_2 + q_2, l_2 \leq l < l_2 + r_2\},$$

$$\left| D^{p_1,q_1,r_1}_{n_1,m_1,l_1}(z') - D^{p_2,q_2,r_2}_{n_2,m_2,l_2}(z') \right| > \frac{1}{s_o}.$$

Therefore, the set of all triple sequences z' contains open neighbourhood of $y \in T$ which does not intersect the set

$$\cap^{\infty}_{\substack{p_1,p_2 = t \\ q_1,q_2 = t \\ r_1,r_2 = t}} \; \cap^{\infty}_{\substack{n_1,n_2 = 1 \\ m_1,m_2 = 1 \\ l_1,l_2 = 1}} \left\{ x \in X : \; \left| D^{p_1,q_1,r_1}_{n_1,m_1,l_1}(x) - D^{p_2,q_2,r_2}_{n_2,m_2,l_2}(x) \right| < \frac{1}{s} \right\} \tag{2}$$

for fixed t and $s = s_o$. Therefore the set (2) is nowhere dense. Hence, the set V is of the first category. This completes the proof. ■

Theorem 2.5 *Almost every $x \in X$ is not almost convergent, i.e.,*

$$R(V) = R(\{x \in X : x = (x_{n,m,l}) \text{ almost convergent }\}) = 0.$$

Proof. Let

$$A^t_s = \{x \in X : x_{n,m,l} = 1, st \leq n < st + s - 1, st \leq m < st + s - 1, st \leq l < st + s - 1\}.$$

The sets A^t_s $(s, t \in \mathbb{N})$ are independent. Since $R(A^t_s) = \frac{1}{2^{s^3}}$ for all $t \in \mathbb{N}$, we have

$$\sum_{t=1}^{\infty} R\left(A^t_s\right) = \sum_{t=1}^{\infty} \frac{1}{2^{s^3}} = \infty.$$

Therefore, due to the second Borel-Cantelli Lemma,

$$R\left(\limsup_{t \to \infty} A^t_s\right) = 1.$$

Let $A_s = \limsup_{t \to \infty} A^t_s$ and $A = \cap^{\infty}_{s=1} A_s$,then $R(A) = 1$. This implies

$$A \setminus \{x \in X : x = (x_{n,m,l}) \text{ almost convergent to } 1\}$$

$$\subset \{x \in X : x = (x_{n,m,l}) \text{ not almost convergent }\}$$

and

$$\{x \in X : x = (x_{n,m,l}) \text{ almost convergent to } 1\}$$

$$\subset \left\{ x \in X : P - \lim_{n,m,l\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} x_{ijk} = 1 \right\}.$$

Since

$$R\left(\left\{ x \in X : P - \lim_{n,m,l\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} x_{ijk} = 1 \right\}\right) = 0$$

it is

$$R\left(\{x \in X : x = (x_{n,m,l}) \text{ not almost convergent }\}\right) = 0.$$

This completes the proof. ∎

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